

THE CONICAL KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA ON FANO MANIFOLD

JIAWEI LIU AND XI ZHANG

ABSTRACT. In this paper, we prove the long-time existence and uniqueness of the conical Kähler-Ricci flow with weak initial data which admits L^p density for some $p > 1$ on Fano manifold. Furthermore, we study the convergence behavior of this flow.

1. INTRODUCTION

Conical Kähler-Einstein metric plays an important role in solving the Yau-Tian-Donaldson's conjecture (see [6, 7, 8, 45]). There has been renewed interest in conical Kähler-Einstein metric in recent years, see references [1, 3, 4, 18, 24, 26, 30, 43] etc. On the other hand, the conical Kähler-Ricci flow was introduced to attack the existence problem of conical Kähler-Einstein metric. The long-time existence and limit behaviour of the conical Kähler-Ricci flow has been widely studied. Chen-Wang [11] studied the strong conical Kähler-Ricci flow and obtained the short-time existence, Y.Q. Wang [48] and the authors [34] got the long-time existence of the conical Kähler-Ricci flow respectively. In [34], the authors also considered the convergence of this flow on Fano manifold with positive twisted first Chern class, they proved that, for any cone angle $0 < 2\pi\beta < 2\pi$, the conical Kähler-Ricci flow converges to a conical Kähler-Einstein metric if there exists one. Chen-Wang [12] obtained the convergence result of this flow when the twisted first Chern class is negative or zero. Later, by adopting the smooth approximation used by the authors in [34], L.M. Shen [38][39] generalized Song-Tian's result [41] and Tian-Zhang's result [46] to the unnormalized conical Kähler-Ricci flow, and G. Edwards [17] obtained the uniform scalar curvature bound when the twisted first Chern class is negative on the foundation of Song-Tian's result [42].

In [34], the authors studied the conical Kähler-Ricci flow which starts with a model conical Kähler metric

$$(1.1) \quad \omega_\beta = \omega_0 + \sqrt{-1}k\partial\bar{\partial}|s|_h^{2\beta}$$

on Fano manifold, where $\omega_0 \in c_1(M)$ is a smooth Kähler metric, s is the defining section of a smooth divisor $D \in |-\lambda K_M|$ and h is a smooth Hermitian metric on $-\lambda K_M$ with curvature $\lambda\omega_0$. In [11, 12, 48], Chen-Wang studied the existence of the conical Kähler-Ricci flow from initial (α, β) metric or weak (α, β) metric with other assumptions.

In this paper, we mainly study the long-time existence, uniqueness and convergence of the conical Kähler-Ricci flow with some weak initial data which admits L^p

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density with $p > 1$ on Fano manifold. We consider the conical Kähler-Ricci flow by using smooth approximation of the twisted Kähler-Ricci flows as that in [34].

Let M be a Fano manifold with complex dimension n , $\omega_0 \in c_1(M)$ be a smooth Kähler metric. For any $p \in (0, \infty]$, we define the class

$$(1.2) \quad \mathcal{E}_p(M, \omega_0) = \left\{ \varphi \in \mathcal{E}(M, \omega_0) \mid \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n} \in L^p(M, \omega_0^n) \right\},$$

where the class

$$(1.3) \quad \mathcal{E}(M, \omega_0) = \left\{ \varphi \in PSH(M, \omega_0) \mid \int_M (\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_M \omega_0^n \right\}$$

defined by Guedj-Zeriahi in [21] is the largest subclass of $PSH(M, \omega_0)$ on which the operator $(\omega_0 + \sqrt{-1}\partial\bar{\partial}\cdot)^n$ is well defined and the comparison principle is valid. When $p > 1$, by S. Kolodziej's L^p -estimate [29] and S. Dinew's uniqueness theorem [14] (see also Theorem B in [21]), we know that the functions in $\mathcal{E}_p(M, \omega_0)$ are Hölder continuous with respect to ω_0 on M .

Let D be a divisor on M . By saying a closed positive $(1, 1)$ -current $\omega \in 2\pi c_1(M)$ with locally bounded potential is a conical Kähler metric with cone angle $2\pi\beta$ ($0 < \beta < 1$) along D , we mean that ω is a smooth Kähler metric on $M \setminus D$, and near each point $p \in D$, there exists local holomorphic coordinate (z^1, \dots, z^n) in a neighborhood U of p such that locally $D = \{z^n = 0\}$, and ω is asymptotically equivalent to the model conical metric

$$(1.4) \quad \sqrt{-1}|z^n|^{2\beta-2}dz^n \wedge d\bar{z}^n + \sqrt{-1}\sum_{j=1}^{n-1} dz^j \wedge d\bar{z}^j \quad \text{on } U.$$

Assume that $D \in |-\lambda K_M|$ ($\lambda \in \mathbb{Q}$), $\mu = 1 - (1 - \beta)\lambda$, $\hat{\omega} \in c_1(M)$ is a Kähler current which admits L^p density with respect to ω_0^n for some $p > 1$ and $\int_M \hat{\omega}^n = \int_M \omega_0^n$. We study the long-time existence, uniqueness and convergence of the following conical Kähler-Ricci flow with weak initial data $\hat{\omega}$

$$(1.5) \quad \begin{cases} \frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) + \mu\omega(t) + (1 - \beta)[D]. \\ \omega(t)|_{t=0} = \hat{\omega} \end{cases}$$

From now on, we denote the Kähler current $\hat{\omega} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0 := \omega_{\varphi_0}$ with $\varphi_0 \in \mathcal{E}_p(M, \omega_0)$ for some $p > 1$.

Definition 1.1. We call $\omega(t)$ a long-time solution to the conical Kähler-Ricci flow (1.5) if it satisfies the following conditions.

- For any $[\delta, T]$ ($\delta, T > 0$), there exist constant C such that

$$C^{-1}\omega_\beta \leq \omega(t) \leq C\omega_\beta \quad \text{on } [\delta, T] \times (M \setminus D);$$

- On $(0, \infty) \times (M \setminus D)$, $\omega(t)$ satisfies the smooth Kähler-Ricci flow;
- On $(0, \infty) \times M$, $\omega(t)$ satisfies equation (1.5) in the sense of currents;
- There exists metric potential $\varphi(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D))$ such that $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ and $\lim_{t \rightarrow 0^+} \|\varphi(t) - \varphi_0\|_{L^\infty(M)} = 0$;
- On $[\delta, T]$, there exist constant $\alpha \in (0, 1)$ and C^* such that the above metric potential $\varphi(t)$ is C^α on M with respect to ω_0 and $\|\frac{\partial \varphi(t)}{\partial t}\|_{L^\infty(M \setminus D)} \leq C^*$.

In Definition 1.1, by saying $\omega(t)$ satisfies equation (1.5) in the sense of currents on $(0, \infty) \times M := M_\infty$, we mean that for any smooth $(n-1, n-1)$ -form $\eta(t)$ with compact support in $(0, \infty) \times M$, we have

$$\int_{M_\infty} \frac{\partial \omega(t)}{\partial t} \wedge \eta(t, x) dt = \int_{M_\infty} (-Ric(\omega(t)) + \mu \omega(t) + (1 - \beta)[D]) \wedge \eta(t, x) dt,$$

where the integral on the left side can be written as

$$\int_{M_\infty} \frac{\partial \omega(t)}{\partial t} \wedge \eta(t, x) dt = - \int_{M_\infty} \omega(t) \wedge \frac{\partial \eta(t, x)}{\partial t} dt$$

in the sense of currents.

We study the conical Kähler-Ricci flow (1.5) by using the following twisted Kähler-Ricci flow with weak initial data ω_{φ_0} .

$$(1.6) \quad \begin{cases} \frac{\partial \omega_\varepsilon(t)}{\partial t} = -Ric(\omega_\varepsilon(t)) + \mu \omega_\varepsilon(t) + \theta_\varepsilon, \\ \omega_\varepsilon(t)|_{t=0} = \omega_{\varphi_0}, \end{cases}$$

where $\theta_\varepsilon = (1 - \beta)(\lambda \omega_0 + \sqrt{-1} \partial \bar{\partial} \log(\varepsilon^2 + |s|_h^2))$ is a smooth closed positive $(1,1)$ -form, s is the definition section of D and h is a smooth Hermitian metric on $-\lambda K_M$ with curvature $\lambda \omega_0$. The smooth case of the twisted Kähler-Ricci flow was studied in [13, 17, 19, 20, 32, 33, 34, 38, 48], etc.

There are some important results on the Kähler-Ricci flow (as well as its twisted versions with smooth twisting form) from weak initial data, such as Chen-Ding [5], Chen-Tian [9], Chen-Tian-Zhang [10], Guedj-Zeriahi [23], Lott-Zhang [35], Nezza-Lu [36], Song-Tian [41], Székelyhidi-Tosatti [44], Z. Zhang [49]. Here, we study the Kähler-Ricci flow which is twisted by non-smooth twisting form, i.e. the flow (1.5). We first obtain the long-time existence, uniqueness and regularity of the flow (1.6) by following Song-Tian's arguments in [41]. Then we study the long-time existence of the conical Kähler-Ricci flow (1.5) by approximating method. In this process, in addition to getting the locally uniform regularity of the twisted Kähler-Ricci flow (1.6), the most important step is to prove that $\varphi(t)$ converges to φ_0 in L^∞ -norm as $t \rightarrow 0^+$ (i.e. the 4th property in Definition 1.1), where $\varphi(t)$ is a metric potential of $\omega(t)$ with respect to metric ω_0 . Here we need a new idea because of Song-Tian's method in [41] is invalid. At the same time, we prove the uniqueness of the conical Kähler-Ricci flow by Jeffres' trick [25] and an improvement of the arguments in [48]. In fact, we obtain the following theorem.

Theorem 1.2. *Let M be a Fano manifold with complex dimension n , $\omega_0 \in c_1(M)$ be a smooth Kähler metric on M , divisor $D \in |-\lambda K_M|$ ($\lambda \in \mathbb{Q}$) and $\hat{\omega} \in c_1(M)$ be a Kähler current which admits L^p density with respect to ω_0^n for some $p > 1$ and $\int_M \hat{\omega}^n = \int_M \omega_0^n$. Then for any $\beta \in (0, 1)$, there exists a unique solution $\omega(t, \cdot)$ to the conical Kähler-Ricci flow (1.5) with weak initial data $\hat{\omega}$.*

Then we consider the convergence of the conical Kähler-Ricci flow (1.5). When $\lambda > 0$ and there is no nontrivial holomorphic field which is tangent to D along D , Tian-Zhu [47] proved a Moser-Trudinger type inequality for conical Kähler-Einstein manifold and gave a new proof of Donaldson's openness theorem [16]. Using the Moser-Trudinger type inequality in [47] and following the arguments in [34], we obtain the following convergence result of the conical Kähler-Ricci flow (1.5).

Theorem 1.3. *Assume that $\lambda > 0$ and there is no nontrivial holomorphic field on M tangent to D , if there exists a conical Kähler-Einstein metric with cone angle $2\pi\beta$ ($0 < \beta < 1$) along D , then the conical Kähler-Ricci flow (1.5) must converge to this conical Kähler-Einstein metric in C_{loc}^∞ topology outside divisor D and globally in the sense of currents on M .*

Remark 1.4. *In this paper, we only study the convergence with positive twisted first Chern class, i.e. $\mu = 1 - (1 - \beta)\lambda > 0$. When $\mu \leq 0$, one can also get the convergence of the conical Kähler-Ricci flow by following Chen-Wang's argument in [12].*

The paper is organized as follows. In section 2, we prove the long-time existence and uniqueness of the twisted Kähler-Ricci flow (1.6) by adopting Song-Tian's methods in [41]. In section 3, we obtain the existence of a long-time solution to the conical Kähler-Ricci flow (1.5) by limiting the twisted Kähler-Ricci flows, and prove that $\varphi(t)$ converges to φ_0 in L^∞ -norm as $t \rightarrow 0^+$, where $\varphi(t)$ is a metric potential of $\omega(t)$ with respect to the metric ω_0 . We also prove the uniqueness of the conical Kähler-Ricci flow with weak initial data ω_{φ_0} . In section 4, by using the uniform Perelman's estimates along the twisted Kähler-Ricci flows obtained in [34], we prove the convergence theorem under the assumptions in Theorem 1.3.

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2. THE LONG-TIME EXISTENCE OF THE TWISTED KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA

In this section, we prove the long-time existence and uniqueness of the twisted Kähler-Ricci flow (1.6) by following Song-Tian's arguments in [41]. For further consideration in the next section, we shall pay attention to the estimates which are independent of ε . In the following arguments, for the sake of brevity, we only consider the flow (1.6) in the case of $\lambda = 1$ (i.e. $\mu = \beta$), where $\beta \in (0, 1)$. Our arguments are also valid for any λ , only if the coefficient β before $\omega_\varepsilon(t)$ in the case of $\lambda = 1$ is replaced by $\mu = 1 - (1 - \beta)\lambda$. We denote

$$F = \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n}{\omega_0^n} \in L^p(M, \omega_0^n) \text{ for } p > 1.$$

Recall that $C^\infty(M)$ is dense in $L^p(M, \omega_0^n)$. Therefore there exists a sequence of positive functions $F_j \in C^\infty(M)$ such that $\int_M F_j \omega_0^n = \int_M \omega_0^n$ and

$$\lim_{j \rightarrow \infty} \|F_j - F\|_{L^p(M)} = 0.$$

By considering the complex Monge-Ampère equation

$$(2.1) \quad (\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{0,j})^n = F_j \omega_0^n$$

and using the stability theorem in [28] (see also [15] or [22]), we have

$$(2.2) \quad \lim_{j \rightarrow \infty} \|\varphi_{0,j} - \varphi_0\|_{L^\infty(M)} = 0,$$

where $\varphi_{0,j} \in PSH(M, \omega_0) \cap C^\infty(M)$ satisfy $\sup_M (\varphi_0 - \varphi_{0,j}) = \sup_M (\varphi_{0,j} - \varphi_0)$.

Let $\omega_{\varphi_{0,j}} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{0,j}$. We prove the long-time existence of the twisted Kähler-Ricci flow (1.6) by using a sequence of smooth twisted Kähler-Ricci flows

$$(2.3) \quad \begin{cases} \frac{\partial \omega_{\varepsilon,j}(t)}{\partial t} = -Ric(\omega_{\varepsilon,j}(t)) + \beta \omega_{\varepsilon,j}(t) + \theta_\varepsilon. \\ \omega_{\varepsilon,j}(t)|_{t=0} = \omega_{\varphi_{0,j}} \end{cases}$$

Since the twisted Kähler-Ricci flow preserves the Kähler class, we can write the flow (2.3) as the parabolic Monge-Ampère equation on potentials,

$$(2.4) \quad \begin{cases} \frac{\partial \varphi_{\varepsilon,j}(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon,j}(t))^n}{\omega_0^n} + F_0 \\ \quad + \beta \varphi_{\varepsilon,j}(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta}, \\ \varphi_{\varepsilon,j}(0) = \varphi_{0,j} \end{cases}$$

where F_0 satisfies $-Ric(\omega_0) + \omega_0 = \sqrt{-1}\partial\bar{\partial}F_0$, $\frac{1}{V} \int_M e^{-F_0} dV_0 = 1$ and $dV_0 = \frac{\omega_0^n}{n!}$. By using the function

$$(2.5) \quad \chi(\varepsilon^2 + |s|_h^2) = \frac{1}{\beta} \int_0^{|\varepsilon^2 + |s|_h^2} \frac{(\varepsilon^2 + r)^\beta - \varepsilon^{2\beta}}{r} dr$$

which was given by Campana-Guenancia-Păun in [4], we can rewrite the flow (2.4) as

$$(2.6) \quad \begin{cases} \frac{\partial \phi_{\varepsilon,j}(t)}{\partial t} = \log \frac{(\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\phi_{\varepsilon,j}(t))^n}{\omega_\varepsilon^n} + F_\varepsilon + \beta(\phi_{\varepsilon,j}(t) + k\chi(\varepsilon^2 + |s|_h^2)), \\ \phi_{\varepsilon,j}(0) = \varphi_{0,j} - k\chi(\varepsilon^2 + |s|_h^2) := \phi_{\varepsilon,0,j} \end{cases}$$

where $\phi_{\varepsilon,j}(t) = \varphi_{\varepsilon,j}(t) - k\chi(\varepsilon^2 + |s|_h^2)$, $\omega_\varepsilon = \omega_0 + \sqrt{-1}k\partial\bar{\partial}\chi(\varepsilon^2 + |s|_h^2)$, $F_\varepsilon = F_0 + \log(\frac{\omega_\varepsilon^n}{\omega_0^n} \cdot (\varepsilon^2 + |s|_h^2)^{1-\beta})$. We know that $\chi(\varepsilon^2 + |s|_h^2)$ and F_ε are uniformly bounded (see (15) and (25) in [4]).

Proposition 2.1. *For any $T > 0$, there exists constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, β , n , ω_0 and T such that for any $t \in [0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$(2.7) \quad \|\phi_{\varepsilon,j}(t)\|_{L^\infty(M)} \leq C.$$

Furthermore, for any j, l , we have

$$(2.8) \quad \|\phi_{\varepsilon,j}(t) - \phi_{\varepsilon,l}(t)\|_{L^\infty([0,T] \times M)} \leq e^{\beta T} \|\varphi_{0,j} - \varphi_{0,l}\|_{L^\infty(M)}.$$

In particular, $\{\varphi_{\varepsilon,j}(t)\}$ satisfies

$$(2.9) \quad \lim_{j,l \rightarrow \infty} \|\varphi_{\varepsilon,j}(t) - \varphi_{\varepsilon,l}(t)\|_{L^\infty([0,T] \times M)} = 0.$$

Proof: From equation (2.6), we have

$$\begin{aligned} \frac{\partial e^{-\beta t} \phi_{\varepsilon,j}(t)}{\partial t} &= e^{-\beta t} \log \frac{(e^{-\beta t} \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}e^{-\beta t} \phi_{\varepsilon,j}(t))^n}{(e^{-\beta t} \omega_\varepsilon)^n} \\ &\quad + e^{-\beta t} (F_\varepsilon + k\beta\chi(\varepsilon^2 + |s|_h^2)) \\ &\leq e^{-\beta t} \log \frac{(e^{-\beta t} \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}e^{-\beta t} \phi_{\varepsilon,j}(t))^n}{(e^{-\beta t} \omega_\varepsilon)^n} + Ce^{-\beta t}, \end{aligned}$$

which is equivalent to

$$\frac{\partial}{\partial t} (e^{-\beta t} (\phi_{\varepsilon,j}(t) + \frac{C}{\beta})) \leq e^{-\beta t} \log \frac{(e^{-\beta t} \omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}e^{-\beta t} (\phi_{\varepsilon,j}(t) + \frac{C}{\beta}))^n}{(e^{-\beta t} \omega_\varepsilon)^n},$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β , n and ω_0 .

For any $\delta > 0$, we denote $\tilde{\phi}_{\varepsilon,j}(t) = e^{-\beta t}(\phi_{\varepsilon,j}(t) + \frac{C}{\beta}) - \delta t$. Let (t_0, x_0) be the maximum point of $\tilde{\phi}_{\varepsilon,j}(t)$ on $[0, T] \times M$. If $t_0 > 0$, by maximum principle, we have

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial t}(e^{-\beta t}(\phi_{\varepsilon,j}(t) + \frac{C}{\beta}))(t_0, x_0) - \delta \\ &\leq e^{-\beta t} \log \frac{(e^{-\beta t} \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_{\varepsilon,j}(t))^n}{(e^{-\beta t} \omega_\varepsilon)^n}(t_0, x_0) - \delta \\ &\leq -\delta, \end{aligned}$$

which is impossible. Hence $t_0 = 0$, then

$$\phi_{\varepsilon,j}(t) \leq e^{\beta t} \sup_M \phi_{\varepsilon,j}(0) + \delta T e^{\beta T} + \frac{C}{\beta}(e^{\beta T} - 1).$$

Let $\delta \rightarrow 0$, we obtain

$$(2.10) \quad \phi_{\varepsilon,j}(t) \leq e^{\beta t} \sup_M \phi_{\varepsilon,j}(0) + \frac{C}{\beta}(e^{\beta T} - 1).$$

By the same arguments, we can get the lower bound of $\phi_{\varepsilon,j}(t)$

$$(2.11) \quad \phi_{\varepsilon,j}(t) \geq e^{\beta t} \inf_M \phi_{\varepsilon,j}(0) - \frac{C}{\beta}(e^{\beta T} - 1).$$

Combining (2.10) and (2.11), we have

$$\|\phi_{\varepsilon,j}(t)\|_{L^\infty(M)} \leq e^{\beta T} \|\phi_{\varepsilon,j}(0)\|_{L^\infty(M)} + \frac{C}{\beta}(e^{\beta T} - 1) \leq C,$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β , n , ω_0 and T .

Let $\psi_{\varepsilon,j,l}(t) = \phi_{\varepsilon,j}(t) - \phi_{\varepsilon,l}(t)$, then $\psi_{\varepsilon,j,l}$ satisfies the following equation

$$(2.12) \quad \begin{cases} \frac{\partial \psi_{\varepsilon,j,l}(t)}{\partial t} = \log \frac{(\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_{\varepsilon,l}(t) + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j,l}(t))^n}{(\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_{\varepsilon,l}(t))^n} + \beta \psi_{\varepsilon,j,l}(t). \\ \psi_{\varepsilon,j,l}(0) = \varphi_{0,j} - \varphi_{0,l} \end{cases}$$

By the same arguments as that in the first part, we have

$$\|\psi_{\varepsilon,j,l}(t)\|_{L^\infty([0,T] \times M)} \leq e^{\beta T} \|\varphi_{0,j} - \varphi_{0,l}\|_{L^\infty(M)}.$$

Since $\{\varphi_{0,j}\}$ is a Cauchy sequence in L^∞ -norm, we conclude the limit (2.9). \square

We now prove the uniform equivalence of the volume forms along the complex Monge-Ampère flow (2.6).

Lemma 2.2. *For any $T > 0$, there exists constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T such that for any $t \in (0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$(2.13) \quad \frac{t^n}{C} \leq \frac{(\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_{\varepsilon,j}(t))^n}{\omega_\varepsilon^n} \leq e^{\frac{C}{t}}.$$

Proof: Let $\Delta_{\varepsilon,j}$ be the Laplacian operator associated to the Kähler form $\omega_{\varepsilon,j}(t) = \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_{\varepsilon,j}(t)$. Straightforward calculations show that

$$(2.14) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right) \dot{\phi}_{\varepsilon,j}(t) = \beta \dot{\phi}_{\varepsilon,j}(t).$$

Let $H_{\varepsilon,j}^+(t) = t\dot{\phi}_{\varepsilon,j}(t) - A\phi_{\varepsilon,j}(t)$, where A is a sufficiently large number (for example $A = \beta T + 2$). Then $H_{\varepsilon,j}^+(0) = -A\phi_{\varepsilon,j}(0)$ is uniformly bounded by a constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, β , n , ω_0 and T .

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)H_{\varepsilon,j}^+(t) &= (1 + \beta t - A)\dot{\phi}_{\varepsilon,j}(t) + A\Delta_{\varepsilon,j}\phi_{\varepsilon,j}(t) \\ (2.15) \qquad \qquad \qquad &\leq (1 + \beta t - A)\dot{\phi}_{\varepsilon,j}(t) + An. \end{aligned}$$

By the maximum principle, $H_{\varepsilon,j}^+(t)$ is uniformly bounded from above by a constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T .

Let $H_{\varepsilon,j}^-(t) = \dot{\phi}_{\varepsilon,j}(t) + \phi_{\varepsilon,j}(t) - n \log t$. Then $H_{\varepsilon,j}^-(t)$ tends to $+\infty$ as $t \rightarrow 0^+$ and

$$(2.16) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)H_{\varepsilon,j}^-(t) = (\beta + 1)\dot{\phi}_{\varepsilon,j}(t) + tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon - n - \frac{n}{t}.$$

Assume that (t_0, x_0) is the minimum point of $H_{\varepsilon,j}^-(t)$ on $[0, T] \times M$. We conclude that $t_0 > 0$ and there exists constant C_1 , C_2 and C_3 such that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)H_{\varepsilon,j}^-(t)|_{(t_0, x_0)} &\geq \left(C_1\left(\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)}\right)^{\frac{1}{n}} + C_2 \log \frac{\omega_{\varepsilon,j}^n(t)}{\omega_\varepsilon^n} - \frac{C_3}{t}\right)|_{(t_0, x_0)} \\ (2.17) \qquad \qquad \qquad &\geq \left(\frac{C_1}{2}\left(\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)}\right)^{\frac{1}{n}} - \frac{C_3}{t}\right)|_{(t_0, x_0)}, \end{aligned}$$

where constant C_1 depends only on n , C_2 depends only on β and C_3 depends only on n , ω_0 , $\|\varphi_0\|_{L^\infty(M)}$, β and T . In inequality (2.17), without loss of generality, we assume that $\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)} > 1$ and $\frac{C_1}{2}\left(\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)}\right)^{\frac{1}{n}} + C_2 \log \frac{\omega_{\varepsilon,j}^n(t)}{\omega_\varepsilon^n} \geq 0$ at (t_0, x_0) . By the maximum principle, we have

$$(2.18) \quad \omega_{\varepsilon,j}^n(t_0, x_0) \geq C_4 t^n \omega_\varepsilon^n(x_0),$$

where C_4 independent of ε and j . Then it easily follows that $H_{\varepsilon,j}^-(t)$ is bounded from below by a constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T . \square

In the following lemma, we prove the uniform equivalence of the metrics along the twisted Kähler-Ricci flow (2.3).

Lemma 2.3. *For any $T > 0$, there exists constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T such that for any $t \in (0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$(2.19) \quad e^{-\frac{C}{t}}\omega_\varepsilon \leq \omega_{\varepsilon,j}(t) \leq e^{\frac{C}{t}}\omega_\varepsilon.$$

Proof: Let

$$(2.20) \quad \Psi_{\varepsilon,\rho} = B \frac{1}{\rho} \int_0^{|s|_h^2} \frac{(\varepsilon^2 + r)^\rho - \varepsilon^{2\rho}}{r} dr$$

be the uniform bound function introduced by Guenancia-Păun in [24]. By choosing suitable B and ρ , and following the arguments in section 2 of [34], we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(t \log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t) + t\Psi_{\varepsilon,\rho}) \\ (2.21) \qquad \qquad \qquad &\leq \log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t) + C tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon + C, \end{aligned}$$

where constant C depends only on n , β , ω_0 and T .

Let $H_{\varepsilon,j}(t) = t \log \operatorname{tr}_{\omega_{\varepsilon}} \omega_{\varepsilon,j}(t) + t \Psi_{\varepsilon,\rho} - A \phi_{\varepsilon,j}(t)$, A be a sufficiently large constant and (t_0, x_0) be the maximum point of $H_{\varepsilon,j}(t)$ on $[0, T] \times M$. We need only consider $t_0 > 0$. By the inequality

$$(2.22) \quad \operatorname{tr}_{\omega_{\varepsilon}} \omega_{\varepsilon,j}(t) \leq \frac{1}{(n-1)!} (\operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon})^{n-1} \frac{\omega_{\varepsilon,j}^n(t)}{\omega_{\varepsilon}^n},$$

we conclude that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right) H_{\varepsilon,j}(t) &\leq \log \operatorname{tr}_{\omega_{\varepsilon}} \omega_{\varepsilon,j}(t) + C \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} - A \dot{\phi}_{\varepsilon,j}(t) - A \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} + C \\ &\leq (n-1) \log \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} - \frac{A}{2} \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} - (A-1) \log \frac{\omega_{\varepsilon,j}^n(t)}{\omega_{\varepsilon}^n} + C, \end{aligned}$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T .

Without loss of generality, we assume that $-\frac{A}{4} \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} + (n-1) \log \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} \leq 0$ at (t_0, x_0) . Then at (t_0, x_0) , by Lemma 2.2, we have

$$(2.23) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right) H_{\varepsilon,j}(t) \leq -\frac{A}{4} \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} - C \log t + C.$$

By the maximum principle, at (t_0, x_0) ,

$$(2.24) \quad \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} \leq C \log \frac{1}{t} + C.$$

By using inequality (2.22), at (t_0, x_0) ,

$$(2.25) \quad \operatorname{tr}_{\omega_{\varepsilon}} \omega_{\varepsilon,j}(t) \leq C \left(\log \frac{1}{t} + 1\right)^{n-1} e^{\frac{C}{t}} \leq e^{\frac{2C}{t}},$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T . Hence we have

$$(2.26) \quad \operatorname{tr}_{\omega_{\varepsilon}} \omega_{\varepsilon,j}(t) \leq e^{\frac{C}{t}}$$

for some constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T .

Furthermore, by inequality (2.22) again, we know

$$(2.27) \quad \operatorname{tr}_{\omega_{\varepsilon,j}(t)} \omega_{\varepsilon} \leq e^{\frac{C}{t}},$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T . From (2.26) and (2.27), we prove the lemma. \square

By Lemma 2.3 and the fact that $\omega_{\varepsilon} > \gamma \omega_0$ for some uniform constant γ (see inequality (24) in [4]), we have

$$(2.28) \quad e^{-\frac{C}{t}} \omega_0 \leq \omega_{\varepsilon,j}(t) \leq C_{\varepsilon} e^{\frac{C}{t}} \omega_0,$$

on $(0, T] \times M$, where C is a uniform constant and C_{ε} depends on ε . We next prove the Calabi's C^3 -estimates. Denote

$$(2.29) \quad S_{\varepsilon,j} = |\nabla_{\omega_0} \omega_{\varepsilon,j}(t)|_{\omega_{\varepsilon,j}(t)}^2 = g_{\varepsilon,j}^{i\bar{m}} g_{\varepsilon,j}^{k\bar{l}} g_{\varepsilon,j}^{p\bar{q}} \nabla_{0i}(g_{\varepsilon,j})_{k\bar{q}} \bar{\nabla}_{0m}(g_{\varepsilon,j})_{p\bar{l}}.$$

Lemma 2.4. *For any $T > 0$ and $\varepsilon > 0$, there exist constants C_{ε} and C such that for any $t \in (0, T]$ and $j \in \mathbb{N}^+$,*

$$(2.30) \quad S_{\varepsilon,j} \leq C_{\varepsilon} e^{\frac{C}{t}},$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T , and constant C_{ε} depends in addition on ε .

Proof: By the similar arguments in [33] or [34] and choosing sufficiently large α and β , we have

$$(2.31) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(e^{-\frac{2\alpha}{t}} S_{\varepsilon,j}) \leq C_\varepsilon e^{-\frac{\alpha}{t}} S_{\varepsilon,j} + C_\varepsilon,$$

$$(2.32) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(e^{-\frac{2\gamma}{t}} \text{tr}_{\omega_0} \omega_{\varepsilon,j}(t)) \leq C_\varepsilon - C_\varepsilon^{-1} e^{-\frac{3\gamma}{t}} S_{\varepsilon,j}.$$

By choosing $A_\varepsilon = C_\varepsilon(C_\varepsilon + 1)$ and $\alpha = 3\gamma$,

$$(2.33) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(e^{-\frac{2\alpha}{t}} S_{\varepsilon,j} + A_\varepsilon e^{-\frac{2\gamma}{t}} \text{tr}_{\omega_0} \omega_{\varepsilon,j}(t)) \leq -e^{-\frac{3\gamma}{t}} S_{\varepsilon,j} + C_\varepsilon.$$

By the maximum principle, we have

$$(2.34) \quad S_{\varepsilon,j} \leq C_\varepsilon e^{\frac{C}{t}} \quad \text{on } (0, T] \times M$$

for some constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , ω_0 and T , and constant C_ε depending in addition on ε . \square

By using the Schauder regularity theory and equation (2.4), we get the high order estimates of $\varphi_{\varepsilon,j}(t)$.

Proposition 2.5. *For any $0 < \delta < T < \infty$, $\varepsilon > 0$ and $k \geq 0$, there exists constant $C_{\varepsilon,\delta,T,k}$ depending only on δ , T , ε , k , n , β , ω_0 and $\|\varphi_0\|_{L^\infty(M)}$, such that for any $j \in \mathbb{N}^+$,*

$$(2.35) \quad \|\varphi_{\varepsilon,j}(t)\|_{C^k([\delta,T] \times M)} \leq C_{\varepsilon,\delta,T,k}.$$

By (2.9), for any $T > 0$, $\varphi_{\varepsilon,j}(t)$ converges to $\varphi_\varepsilon(t) \in L^\infty([0, T] \times M)$ uniformly in $L^\infty([0, T] \times M)$. For any $0 < \delta < T < \infty$ and $\varepsilon > 0$, $\varphi_{\varepsilon,j}(t)$ is uniformly bounded (depends on ε) in $C^\infty([\delta, T] \times M)$. Therefore $\varphi_{\varepsilon,j}(t)$ converges to $\varphi_\varepsilon(t)$ in $C^\infty([\delta, T] \times M)$. Hence for any $\varepsilon > 0$, $\varphi_\varepsilon(t) \in C^\infty((0, \infty) \times M)$.

Proposition 2.6. *For any $\varepsilon > 0$, $\varphi_\varepsilon(t) \in C^0([0, \infty) \times M)$ and*

$$(2.36) \quad \lim_{t \rightarrow 0^+} \|\varphi_\varepsilon(t) - \varphi_0\|_{L^\infty(M)} = 0.$$

Proof: For any $(t, z) \in (0, T] \times M$,

$$(2.37) \quad \begin{aligned} |\varphi_\varepsilon(t, z) - \varphi_0(z)| &\leq |\varphi_\varepsilon(t, z) - \varphi_{\varepsilon,j}(t, z)| + |\varphi_{\varepsilon,j}(t, z) - \varphi_{0,j}(z)| \\ &\quad + |\varphi_{0,j}(z) - \varphi_0(z)|. \end{aligned}$$

Since $\varphi_{\varepsilon,j}(t)$ is a Cauchy sequence in $L^\infty([0, T] \times M)$,

$$(2.38) \quad \lim_{j \rightarrow \infty} \|\varphi_\varepsilon(t, z) - \varphi_{\varepsilon,j}(t, z)\|_{L^\infty([0, T] \times M)} = 0.$$

From (2.2), we have

$$(2.39) \quad \lim_{j \rightarrow \infty} \|\varphi_{0,j}(z) - \varphi_0(z)\|_{L^\infty(M)} = 0,$$

For any $\epsilon > 0$, there exists N such that for any $j > N$,

$$\begin{aligned} \sup_{[0, T] \times M} |\varphi_\varepsilon(t, z) - \varphi_{\varepsilon,j}(t, z)| &< \frac{\epsilon}{3}, \\ \sup_M |\varphi_{0,j}(z) - \varphi_0(z)| &< \frac{\epsilon}{3}. \end{aligned}$$

On the other hand, fix such j , there exists $0 < \delta < T$ such that

$$(2.40) \quad \sup_{[0, \delta] \times M} |\varphi_{\varepsilon, j}(t, z) - \varphi_{0, j}| < \frac{\epsilon}{3}.$$

Combining the above estimates together, for any $t \in [0, \delta]$ and $z \in M$,

$$(2.41) \quad |\varphi_{\varepsilon}(t, z) - \varphi_0(z)| < \epsilon.$$

This completes the proof of the lemma. \square

Proposition 2.7. $\varphi_{\varepsilon}(t)$ is the unique solution to the parabolic Monge-Ampère equation

$$(2.42) \quad \begin{cases} \frac{\partial \varphi_{\varepsilon}(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon}(t))^n}{\omega_0^n} + F_0 \\ \quad + \beta \varphi_{\varepsilon}(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta}, & (0, \infty) \times M \\ \varphi_{\varepsilon}(0) = \varphi_0 \end{cases}$$

in the space of $C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M)$.

Proof: By proposition 2.6, we only need to prove the uniqueness. Suppose there exists another solution $\tilde{\varphi}_{\varepsilon}(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M)$ to the Monge-Ampère equation (2.42).

Let $\psi_{\varepsilon}(t) = \tilde{\varphi}_{\varepsilon}(t) - \varphi_{\varepsilon}(t)$. Then

$$(2.43) \quad \begin{cases} \frac{\partial \psi_{\varepsilon}(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon}(t) + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon}(t))^n}{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon}(t))^n} + \beta \psi_{\varepsilon}(t). \\ \psi_{\varepsilon}(0) = 0 \end{cases}$$

For any $T > 0$, by the same arguments as that in the proof of Proposition 2.1, we have

$$\|\psi_{\varepsilon}(t)\|_{L^\infty([0, T] \times M)} \leq e^{\beta T} \|\psi_{\varepsilon}(0)\|_{L^\infty(M)} = 0.$$

Hence $\psi_{\varepsilon}(t) = 0$, that is $\tilde{\varphi}_{\varepsilon}(t) = \varphi_{\varepsilon}(t)$. \square

By the similar arguments as that in [41], we prove the uniqueness theorems of the twisted Kähler-Ricci flow.

Theorem 2.8. Let M be a Fano manifold with complex dimension n , $\omega_0 \in c_1(M)$ be a smooth Kähler metric on M and $\hat{\omega} \in c_1(M)$ be a Kähler current which admits L^p density with respect to ω_0^n for some $p > 1$ and $\int_M \hat{\omega}^n = \int_M \omega_0^n$. Then there exists a unique solution $\omega_{\varepsilon}(t) \in C^\infty((0, \infty) \times M)$ to the twisted Kähler-Ricci flow (1.6) with initial data $\hat{\omega}$ in the following sense.

$$(1) \quad \frac{\partial \omega_{\varepsilon}(t)}{\partial t} = -\text{Ric}(\omega_{\varepsilon}(t)) + \beta \omega_{\varepsilon}(t) + \theta_{\varepsilon} \text{ on } (0, \infty) \times M;$$

(2) There exists $\varphi_{\varepsilon}(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M)$ such that $\omega_{\varepsilon}(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\varepsilon}(t)$ and

$$(2.44) \quad \lim_{t \rightarrow 0^+} \|\varphi_{\varepsilon}(t) - \varphi_0\|_{L^\infty(M)} = 0,$$

where $\varphi_0 \in \mathcal{E}_p(M, \omega_0)$ is a metric potential of $\hat{\omega}$ with respect to ω_0 . In particular, $\omega_{\varepsilon}(t)$ converges in the sense of distribution to $\hat{\omega}$ as $t \rightarrow 0$.

Proof: From Proposition 2.7, we know that there exists a solution $\omega_\varepsilon(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t)$ to the twisted Kähler-Ricci flow (1.6) with initial data $\hat{\omega}$, where $\varphi_\varepsilon(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M)$ satisfies

$$(2.45) \quad \lim_{t \rightarrow 0^+} \|\varphi_\varepsilon(t) - \varphi_0\|_{L^\infty(M)} = 0$$

for some metric potential $\varphi_0 \in \mathcal{E}_p(M, \omega_0)$ of $\hat{\omega}$ with respect to ω_0 . Suppose that there is another solution $\tilde{\omega}_\varepsilon(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\varepsilon(t)$ to the twisted Kähler-Ricci flow (1.6) with initial data $\hat{\omega}$. Then $\tilde{\varphi}_\varepsilon(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M)$ satisfies

$$(2.46) \quad \begin{aligned} \frac{\partial \tilde{\varphi}_\varepsilon(t)}{\partial t} &= \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\varepsilon(t))^n}{\omega_0^n} + F_0 \\ &\quad + \beta \tilde{\varphi}_\varepsilon(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta} + f_\varepsilon(t) \end{aligned}$$

on $(0, \infty) \times M$ for a smooth function $f_\varepsilon(t)$ on $(0, \infty)$ and

$$\lim_{t \rightarrow 0^+} \|\tilde{\varphi}_\varepsilon(t) - \tilde{\varphi}_0\|_{L^\infty(M)} = 0,$$

where $\tilde{\varphi}_0 \in \mathcal{E}_p(M, \omega_0)$ is also a metric potential of $\hat{\omega}$ with respect to ω_0 . At the same time, we have $\varphi_0 = \tilde{\varphi}_0 + \tilde{C}$.

Let $\hat{\varphi}(t) = \tilde{\varphi}(t) + \tilde{C}e^{\beta t}$. It is obvious that $\hat{\varphi}_\varepsilon(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M)$ is a solution to equation (2.46) and satisfies

$$\lim_{t \rightarrow 0^+} \|\hat{\varphi}_\varepsilon(t) - \varphi_0\|_{L^\infty(M)} = 0.$$

Now we consider the function $\psi_\varepsilon(t) = \hat{\varphi}_\varepsilon(t) - \varphi_\varepsilon(t)$.

$$(2.47) \quad \begin{cases} \frac{\partial \psi_\varepsilon(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t) + \sqrt{-1}\partial\bar{\partial}\psi_\varepsilon(t))^n}{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t))^n} + \beta \psi_\varepsilon(t) + f_\varepsilon(t). \\ \psi_\varepsilon(0) = 0 \end{cases}$$

For any $0 < t_1 < t_2 < \infty$, by the same arguments as that in the proof of Proposition 2.1, we have

$$\begin{aligned} \sup_M \psi_\varepsilon(t_2) &\leq e^{\beta(t_2-t_1)} \sup_M \psi_\varepsilon(t_1) + \int_{t_1}^{t_2} e^{\beta(t_2-t)} f_\varepsilon(t) dt, \\ \inf_M \psi_\varepsilon(t_2) &\geq e^{\beta(t_2-t_1)} \inf_M \psi_\varepsilon(t_1) + \int_{t_1}^{t_2} e^{\beta(t_2-t)} f_\varepsilon(t) dt. \end{aligned}$$

Therefore, we obtain

$$\inf_M \psi_\varepsilon(t_2) \geq \sup_M \psi_\varepsilon(t_2) - e^{\beta(t_2-t_1)} (\sup_M \psi_\varepsilon(t_1) - \inf_M \psi_\varepsilon(t_1)).$$

Let $t_1 \rightarrow 0^+$, we have

$$\inf_M \psi_\varepsilon(t_2) \geq \sup_M \psi_\varepsilon(t_2).$$

By equation (2.47), $\psi_\varepsilon(t) = \int_0^t e^{\beta(t-s)} f_\varepsilon(s) ds$. Hence $\tilde{\omega}_\varepsilon(t) = \omega_\varepsilon(t)$. \square

3. THE LONG-TIME EXISTENCE OF THE CONICAL KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA

In this section, we study the long-time existence of the conical Kähler-Ricci flow (1.5) by the smooth approximation of the twisted Kähler-Ricci flows. We also prove the uniqueness of the conical Kähler-Ricci flow (1.5).

By Proposition 2.1, Lemma 2.3 and Proposition 2.5, we conclude that for any $T > 0$, there exists constants C_1 and C_2 depending only on $\|\varphi_0\|_{L^\infty(M)}$, β , n , ω_0 and T , such that for any $\varepsilon > 0$,

$$(3.1) \quad \|\phi_\varepsilon(t)\|_{L^\infty([0,T] \times M)} \leq C_1,$$

$$(3.2) \quad e^{-\frac{C_2}{t}} \omega_\varepsilon \leq \omega_\varepsilon(t) \leq e^{\frac{C_2}{t}} \omega_\varepsilon \quad \text{on } (0, T] \times M.$$

We first prove the local uniform Calabi's C^3 -estimate and curvature estimate along the flow (2.3). Our proofs are similar as that in [34] (see section 2 in [34] or section 3 in [40]), but we need some arguments to handle the weak initial data case.

Lemma 3.1. *For any $T > 0$ and $B_r(p) \subset \subset M \setminus D$, there exist constants C , C' and C'' such that for any $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$\begin{aligned} S_{\varepsilon,j} &\leq \frac{C'}{r^2} e^{\frac{C}{t}}, \\ |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 &\leq \frac{C''}{r^4} e^{\frac{C}{t}} \end{aligned}$$

on $(0, T] \times B_{\frac{r}{2}}(p)$, where constants C , C' and C'' depend only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , T , ω_0 and $\text{dist}_{\omega_0}(B_r(p), D)$.

Proof: By Lemma 2.3, there exists uniform constat C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , T , ω_0 and $\text{dist}_{\omega_0}(B_r(p), D)$, such that

$$(3.3) \quad e^{-\frac{C}{t}} \omega_0 \leq \omega_{\varepsilon,j}(t) \leq e^{\frac{C}{t}} \omega_0, \quad \text{on } B_r(p) \times (0, T].$$

Let $r = r_0 > r_1 > \frac{r}{2}$ and ψ be a nonnegative C^∞ cut-off function that is identically equal to 1 on $\overline{B_{r_1}(p)}$ and vanishes outside $B_r(p)$. We may assume that

$$(3.4) \quad |\partial\psi|_{\omega_0}^2 \leq \frac{C}{r^2} \quad \text{and} \quad |\sqrt{-1}\partial\bar{\partial}\psi|_{\omega_0} \leq \frac{C}{r^2}.$$

Straightforward calculations show that

$$(3.5) \quad \left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(\psi^2 S_{\varepsilon,j}) \leq \frac{C}{r^2} e^{\frac{C}{t}} S_{\varepsilon,j} + C e^{\frac{C}{t}}.$$

By choosing sufficiently large α , γ and A , we get

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(e^{-\frac{2\alpha}{t}} \psi^2 S_{\varepsilon,j} + A e^{-\frac{2\gamma}{t}} \text{tr}_{\omega_0} \omega_{\varepsilon,j}(t)) \\ &\leq -\frac{1}{r^2} e^{-\frac{3\gamma}{t}} S_{\varepsilon,j} + \frac{C}{r^2}, \end{aligned}$$

where $\alpha = 3\gamma$, $A = \frac{C+1}{r^2}$, constat C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , T , ω_0 and $\text{dist}_{\omega_0}(B_r(p), D)$. By the maximum principle, we conclude that

$$S_{\varepsilon,j} \leq \frac{C'}{r^2} e^{\frac{6\gamma}{t}} \quad \text{on } (0, T] \times B_{\frac{r}{2}}(p).$$

Now we prove that $|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2$ is uniformly bounded. Through computation, there exist uniform constants C such that

$$\begin{aligned} & \left(\frac{d}{dt} - \Delta_{\varepsilon,j}\right)|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \\ & \leq C|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^3 + Ce^{\frac{C}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 + Ce^{\frac{C}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)} + Ce^{\frac{C}{t}}S_{\varepsilon,j}^{\frac{1}{2}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)} \\ & \quad + Ce^{\frac{C}{t}}S_{\varepsilon,j}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)} - |\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 - |\bar{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 + Ce^{\frac{C}{t}} \\ & \leq C(|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^3 + e^{\frac{C}{t}} + \frac{1}{r^2}e^{\frac{C}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}) - |\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 - |\bar{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2. \end{aligned}$$

Next, we show that $|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2$ is uniformly bounded. We fix a smaller radius r_2 satisfying $r_1 > r_2 > \frac{r}{2}$. Let ρ be a cut-off function identically equal to 1 on $\overline{B_{r_2}}(p)$ and identically equal to 0 outside B_{r_1} . We also let ρ satisfy

$$|\partial\rho|_{\omega_0}^2, |\sqrt{-1}\partial\bar{\partial}\rho|_{\omega_0} \leq \frac{C}{r^2}$$

for some uniform constant C . From the former part we know that $S_{\varepsilon,j}$ is bounded by $\frac{C}{r^2}e^{\frac{C}{t}}$ on $B_{r_1}(p)$. Let $K_t = \frac{\hat{C}}{r^2}e^{\frac{k\tau}{t}}$, k and \hat{C} be constants which are large enough such that $\frac{K_t}{2} \leq K_t - S_{\varepsilon,j} \leq K_t$. We consider

$$(3.6) \quad F_{\varepsilon,j} = \rho^2 e^{-\frac{2\delta}{t}} \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}} + Ae^{-\frac{2\sigma}{t}} S_{\varepsilon,j}.$$

By computing, we have

$$\begin{aligned} & \left(\frac{d}{dt} - \Delta_{\varepsilon,j}\right)F_{\varepsilon,j} \\ & = e^{-\frac{2\delta}{t}} \left((-\Delta_{\varepsilon,j}\rho^2) \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{(K_t - S_{\varepsilon,j})^2} \left(\frac{d}{dt} - \Delta_{\varepsilon,j}\right)S_{\varepsilon,j} + \rho^2 \frac{\hat{C}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{r^2(K_t - S_{\varepsilon,j})^2} \frac{k\tau}{t^2} e^{\frac{k\tau}{t}} \right. \\ & \quad + \rho^2 \frac{1}{K_t - S_{\varepsilon,j}} \left(\frac{d}{dt} - \Delta_{\varepsilon,j}\right)|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 - 4Re\langle \rho \frac{\nabla_{\varepsilon,j}\rho}{K_t - S_{\varepsilon,j}}, \nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \rangle_{\omega_{\varepsilon,j}(t)} \\ & \quad - 4Re\langle \rho \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{(K_t - S_{\varepsilon,j})^2} \nabla_{\varepsilon,j}S_{\varepsilon,j}, \nabla_{\varepsilon,j}\rho \rangle_{\omega_{\varepsilon,j}(t)} - 2 \frac{\rho^2 |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{(K_t - S_{\varepsilon,j})^3} |\nabla_{\varepsilon,j}S|_{\omega_{\varepsilon,j}(t)}^2 \\ & \quad \left. - 2Re\langle \rho^2 \frac{\nabla_{\varepsilon,j}S_{\varepsilon,j}}{(K_t - S_{\varepsilon,j})^2}, \nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \rangle_{\omega_{\varepsilon,j}(t)} \right) + Ae^{-\frac{2\sigma}{t}} \left(\frac{d}{dt} - \Delta_{\omega_{\varepsilon,j}(t)}\right)S_{\varepsilon,j} \\ & \quad + A \frac{2\sigma}{t^2} e^{-\frac{2\sigma}{t}} S_{\varepsilon,j} + \frac{2\delta}{t^2} e^{-\frac{2\delta}{t}} \rho^2 \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}}. \end{aligned}$$

We only consider an inner point (t_0, x_0) which is a maximum point of $F_{\varepsilon,j}$ achieved on $[0, T] \times \overline{B_{r_1}}(p)$. We use the fact that $\nabla_{\varepsilon,j}F_{\varepsilon,j} = 0$ at this point,

$$\begin{aligned} & e^{-\frac{2\delta}{t}} \left(2\rho \nabla_{\varepsilon,j}\rho \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{\nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \nabla_{\varepsilon,j}S}{(K_t - S_{\varepsilon,j})^2} \right) \\ & + Ae^{-\frac{2\sigma}{t}} \nabla_{\varepsilon,j}S_{\varepsilon,j} = 0. \end{aligned}$$

Combining the above two equalities, we have

$$\begin{aligned}
& \left(\frac{d}{dt} - \Delta_{\varepsilon,j}\right)F_{\varepsilon,j} \\
= & e^{-\frac{2\delta}{t}} \left((-\Delta_{\varepsilon,j}\rho^2) \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{(K_t - S_{\varepsilon,j})^2} \left(\frac{d}{dt} - \Delta_{\varepsilon,j} \right) S_{\varepsilon,j} + \rho^2 \frac{\hat{C}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{r^2(K_t - S_{\varepsilon,j})^2} \frac{k\tau}{t^2} e^{\frac{k\tau}{t}} \right. \\
& + \rho^2 \frac{1}{K_t - S_{\varepsilon,j}} \left(\frac{d}{dt} - \Delta_{\varepsilon,j} \right) |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 - 4Re \left\langle \rho \frac{\nabla_{\varepsilon,j}\rho}{K_t - S_{\varepsilon,j}}, \nabla_{\varepsilon,j} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \right\rangle_{\omega_{\varepsilon,j}(t)} \Big) \\
& + 2Ae^{-\frac{2\sigma}{t}} \frac{|\nabla_{\varepsilon,j}S_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}} + Ae^{-\frac{2\sigma}{t}} \left(\frac{d}{dt} - \Delta_{\omega_{\varepsilon,j}(t)} \right) S_{\varepsilon,j} + A \frac{2\sigma}{t^2} e^{-\frac{2\sigma}{t}} S_{\varepsilon,j} \\
& + \frac{2\delta}{t^2} e^{-\frac{2\delta}{t}} \rho^2 \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t - S_{\varepsilon,j}}.
\end{aligned}$$

Our goal is to show that at (t_0, x_0) we have $e^{-\frac{2\delta}{t}} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \leq \frac{C}{r^4}$ for some uniform constant C and δ . Without loss of generality, we assume that $|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^3 \geq e^{\frac{\tau}{t}} + \frac{1}{r^2} e^{\frac{\tau}{t}} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}$ at (t_0, x_0) .

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{\omega_{\varepsilon,j}(t)}\right) |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 & \leq C |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^3 - |\nabla_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 - |\overline{\nabla}_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2, \\
\left(\frac{d}{dt} - \Delta_{\omega_{\varepsilon,j}(t)}\right) S_{\varepsilon,j} & \leq \frac{C}{r^2} e^{\frac{\tau}{t}} - |\nabla_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2 - |\overline{\nabla}_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2
\end{aligned}$$

on $B_{r_1}(p)$. We also note that

$$\begin{aligned}
|\nabla_{\varepsilon,j} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2|_{\omega_{\varepsilon,j}(t)} & \leq |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)} (|\nabla_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)} + |\overline{\nabla}_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}), \\
|\nabla_{\varepsilon,j} S_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 & \leq 2S_{\varepsilon,j} (|\nabla_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2).
\end{aligned}$$

By using the above inequalities, at (t_0, x_0) , we have

$$\begin{aligned}
& \left(\frac{d}{dt} - \Delta_{\omega_{\varepsilon,j}(t)}\right)F_{\varepsilon,j} \\
\leq & -Ae^{-\frac{2\sigma}{t}} (|\nabla_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2) + \frac{AC}{r^2} e^{-\frac{2\sigma}{t}} e^{\frac{\tau}{t}} + e^{-\frac{2\delta}{t}} \left(\frac{Ce^{\frac{\tau}{t}} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t r^2} \right. \\
& - \frac{\rho^2 |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 (|\nabla_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2)}{K_t^2} + \frac{C\rho^2 |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^3}{K_t} \\
& - \frac{\rho^2 (|\nabla_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2)}{K_t} + \frac{Ce^{\frac{\tau}{t}} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t r^2} + \frac{C\rho^2 |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t^2 r^2} e^{\frac{\tau}{t}} \\
& + \frac{\rho^2 (|\nabla_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2)}{K_t} \Big) + \frac{8Ae^{-\frac{2\sigma}{t}} S_{\varepsilon,j} (|\nabla_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2)}{K_t} \\
& + \rho^2 e^{-\frac{2\delta}{t}} \frac{2\hat{C}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{r^2 K_t^2} \frac{k\tau}{t^2} e^{\frac{k\tau}{t}} + A \frac{2\sigma}{t^2} e^{-\frac{2\sigma}{t}} S_{\varepsilon,j} + \frac{4\delta}{t^2} e^{-\frac{2\delta}{t}} \rho^2 \frac{|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2}{K_t}.
\end{aligned}$$

Let \hat{C} be sufficiently large so that $\frac{8AS_{\varepsilon,j}Q}{K_t} \leq \frac{AQ}{2}$, where we denote $Q = |\nabla_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2 + |\overline{\nabla}_{\varepsilon,j} X|_{\omega_{\varepsilon,j}(t)}^2$. Then

$$\begin{aligned}
(3.7) \quad \frac{C\rho^2|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^3}{K_t} &\leq \frac{\rho^2|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^4}{2K_t^2} + C\rho^2|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \\
&\leq \frac{\rho^2|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 Q}{K_t^2} + Ce^{\frac{C}{t}}\rho^2|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2
\end{aligned}$$

Let $k = 1$, $\delta = 2\sigma$ and $\tau - 2\sigma < 0$, where σ is sufficiently large. We conclude that the evolution equation of $F_{\varepsilon,j}$ can be controlled as follows,

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{\varepsilon,j}\right)F_{\varepsilon,j} &\leq -\frac{Ae^{-\frac{2\sigma}{t}}Q}{2} + \frac{AC}{r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}} + \frac{AC}{t^2r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}} + Ce^{-\frac{\delta}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \\
&\leq -\frac{Ae^{-\frac{2\sigma}{t}}Q}{2} + \frac{AC}{r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}} + \frac{AC}{t^2r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}} + Ce^{-\frac{\delta}{t}}Q + Ce^{-\frac{\delta}{2t}}.
\end{aligned}$$

Now we choose a sufficiently large A such that $A = 2(C + 1)$ and obtain

$$e^{-\frac{\delta}{t}}Q \leq \frac{C}{r^2}$$

at (t_0, x_0) . This implies that $e^{-\frac{2\delta}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \leq \frac{C}{r^2}$ at this point, where C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , T , $\text{dist}_{\omega_0}(B_r(p), D)$, $\|\theta\|_{C^2(B_r(p))}$ and ω_0 . Following that we conclude that $F_{\varepsilon,j}$ is bounded by $\frac{C}{r^2}$ at (t_0, x_0) . Hence on $[0, T] \times \overline{B_{r_2}(p)}$, we obtain

$$(3.8) \quad |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \leq \frac{C}{r^4}e^{\frac{2\delta+\tau}{t}},$$

where C , δ and τ depend only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , T , $\text{dist}_{\omega_0}(B_r(p), D)$ and ω_0 . \square

By using the standard parabolic Schauder regularity theory [31], we obtain the following proposition.

Proposition 3.2. *For any $0 < \delta < T < \infty$, $k \in \mathbb{N}^+$ and $B_r(p) \subset\subset M \setminus D$, there exists constant $C_{\delta,T,k,p,r}$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , δ , k , T , $\text{dist}_{\omega_0}(B_r(p), D)$ and ω_0 , such that for any $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$(3.9) \quad \|\varphi_{\varepsilon,j}(t)\|_{C^k([\delta,T] \times B_r(p))} \leq C_{\delta,T,k,p,r}.$$

Through a further observation to equation (2.42), we prove the monotonicity of $\varphi_\varepsilon(t)$ with respect to ε .

Proposition 3.3. *For any $(t, x) \in [0, T] \times M$, $\varphi_\varepsilon(t, x)$ is monotone decreasing as $\varepsilon \searrow 0$.*

Proof: For any $\varepsilon_1 < \varepsilon_2$, let $\psi_{1,2}(t) = \varphi_{\varepsilon_1}(t) - \varphi_{\varepsilon_2}(t)$. Then we have

$$\begin{aligned}
(3.10) \quad &\frac{\partial}{\partial t}(e^{-\beta t}\psi_{1,2}(t)) \\
&\leq e^{-\beta t} \log \frac{(e^{-\beta t}\omega_0 + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\varphi_{\varepsilon_2}(t) + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\psi_{1,2}(t))^n}{(e^{-\beta t}\omega_0 + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\varphi_{\varepsilon_2}(t))^n}.
\end{aligned}$$

Let $\tilde{\psi}_{1,2}(t) = e^{-\beta t}\psi_{1,2}(t) - \delta t$ with $\delta > 0$ and (t_0, x_0) be the maximum point of $\tilde{\psi}_{1,2}(t)$ on $[0, T] \times M$. If $t_0 > 0$, by maximum principle, at this point, we have

$$(3.11) \quad 0 \leq \frac{\partial}{\partial t}\tilde{\psi}_{1,2}(t) = \frac{\partial}{\partial t}(e^{-\beta t}\psi_{1,2}(t)) - \delta \leq -\delta$$

which is impossible, hence $t_0 = 0$. So for any $(t, x) \in [0, T] \times M$,

$$(3.12) \quad \psi_{1,2}(t, x) \leq e^{\beta t} \sup_M \psi_{1,2}(0, x) + T e^{\beta T} \delta = T e^{\beta T} \delta.$$

Let $\delta \rightarrow 0$, we conclude that $\varphi_{\varepsilon_1}(t, x) \leq \varphi_{\varepsilon_2}(t, x)$. \square

For any $[\delta, T] \times K \subset (0, \infty) \times M \setminus D$ and $k \geq 0$, $\|\varphi_{\varepsilon,j}(t)\|_{C^k([\delta,T] \times K)}$ is uniformly bounded by Proposition 3.2. Let j approximate to ∞ , we obtain that $\|\varphi_{\varepsilon}(t)\|_{C^k([\delta,T] \times K)}$ is uniformly bounded. Then let δ approximate to 0, T approximate to ∞ and K approximate to $M \setminus D$, by diagonal rule, we get a sequence $\{\varepsilon_i\}$, such that $\varphi_{\varepsilon_i}(t)$ converges in C_{loc}^{∞} topology on $(0, \infty) \times (M \setminus D)$ to a function $\varphi(t)$ that is smooth on $C^{\infty}((0, \infty) \times (M \setminus D))$ and satisfies equation

$$(3.13) \quad \frac{\partial \varphi(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n}{\omega_0^n} + F_0 + \beta \varphi(t) + \log |s|_h^{2(1-\beta)}$$

on $(0, \infty) \times (M \setminus D)$. Since $\varphi_{\varepsilon}(t)$ is monotone decreasing as $\varepsilon \rightarrow 0$, we conclude that $\varphi_{\varepsilon}(t)$ converges in C_{loc}^{∞} topology on $(0, \infty) \times (M \setminus D)$ to $\varphi(t)$. Combining the above arguments with (3.1) and (3.2), for any $T > 0$, we have

$$(3.14) \quad \|\varphi(t)\|_{L^{\infty}((0,T] \times (M \setminus D))} \leq C_1,$$

$$(3.15) \quad e^{-\frac{C_2}{t}} \omega_{\beta} \leq \omega(t) \leq e^{\frac{C_2}{t}} \omega_{\beta} \quad \text{on } (0, T] \times (M \setminus D),$$

where $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)$, constants C_1 and C_2 depend only on $\|\varphi_0\|_{L^{\infty}(M)}$, β , n , ω_0 and T .

Proposition 3.4. *For any $t > 0$, $\varphi(t)$ is Hölder continuous on M with respect to the metric ω_0 .*

Proof: We assume that $t \in [\delta, T]$ for some δ and T satisfying $0 < \delta < T < \infty$. By (3.15) we have

$$(3.16) \quad C^{-1} \omega_{\beta} \leq \omega(t) \leq C \omega_{\beta} \quad \text{on } [\delta, T] \times (M \setminus D),$$

where constant C depends only on $\|\varphi_0\|_{L^{\infty}(M)}$, β , T , n , ω_0 and δ . Combining this estimate and the fact that $\log \frac{\omega_{\beta}^n |s|_h^{2(1-\beta)}}{\omega_0^n}$ is bounded uniformly on $M \setminus D$, we obtain

$$(3.17) \quad \left\| \log \frac{\omega^n(t) |s|_h^{2(1-\beta)}}{\omega_0^n} \right\|_{L^{\infty}([\delta, T] \times (M \setminus D))} \leq C$$

for some uniform constant C independent of t . Therefore, $\left\| \frac{\partial \varphi(t)}{\partial t} \right\|_{L^{\infty}([\delta, T] \times (M \setminus D))}$ is uniformly bounded by equation (3.13) and estimate (3.14). We rewrite equation (3.13) as

$$(3.18) \quad (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n = e^{\frac{\partial \varphi(t)}{\partial t} - F_0 - \beta \varphi(t)} \frac{\omega_0^n}{|s|_h^{2(1-\beta)}}.$$

The function on the right side of equation (3.18) is L^p integrable with respect to ω_0^n for some $p > 1$. By S. Kolodziej's L^p -estimates [29], we know that $\varphi(t)$ is Hölder continuous on M with respect to ω_0 for any $t > 0$. \square

Next, by using the monotonicity of $\varphi_{\varepsilon}(t)$ with respect to ε and constructing auxiliary function, we prove the continuity of $\varphi(t)$ as $t \rightarrow 0^+$.

Proposition 3.5. $\varphi(t) \in C^0([0, \infty) \times M)$ and

$$(3.19) \quad \lim_{t \rightarrow 0^+} \|\varphi(t) - \varphi_0\|_{L^\infty(M)} = 0.$$

Proof: Through the above arguments, we only need prove limit (3.19). By the monotonicity of $\varphi_\varepsilon(t)$ with respect to ε , for any $(t, z) \in (0, T] \times M$, we have

$$(3.20) \quad \begin{aligned} \varphi(t, z) - \varphi_0(z) &\leq \varphi_{\varepsilon_1}(t, z) - \varphi_0(z) \\ &\leq |\varphi_{\varepsilon_1}(t, z) - \varphi_{\varepsilon_1, j}(t, z)| + |\varphi_{\varepsilon_1, j}(t, z) - \varphi_{0, j}(z)| \\ &\quad + |\varphi_{0, j}(z) - \varphi_0(z)|. \end{aligned}$$

Since $\varphi_{\varepsilon_1, j}(t)$ is a Cauchy sequence in $L^\infty([0, T] \times M)$,

$$(3.21) \quad \lim_{j \rightarrow \infty} \|\varphi_{\varepsilon_1}(t, z) - \varphi_{\varepsilon_1, j}(t, z)\|_{L^\infty([0, T] \times M)} = 0.$$

From (2.2), we have

$$(3.22) \quad \lim_{j \rightarrow \infty} \|\varphi_{0, j}(z) - \varphi_0(z)\|_{L^\infty(M)} = 0,$$

For any $\epsilon > 0$, there exists N such that for any $j > N$,

$$\begin{aligned} \sup_{[0, T] \times M} |\varphi_{\varepsilon_1}(t, z) - \varphi_{\varepsilon_1, j}(t, z)| &< \frac{\epsilon}{3}, \\ \sup_M |\varphi_{0, j}(z) - \varphi_0(z)| &< \frac{\epsilon}{3}. \end{aligned}$$

Fix such ε_1 and j , there exists $0 < \delta_1 < T$ such that

$$(3.23) \quad \sup_{[0, \delta_1] \times M} |\varphi_{\varepsilon_1, j}(t, z) - \varphi_{0, j}(z)| < \frac{\epsilon}{3}.$$

Combining the above estimates together, for any $t \in (0, \delta_1]$ and $z \in M$,

$$(3.24) \quad \varphi(t, z) - \varphi_0(z) < \epsilon.$$

On the other hand, by S. Kolodziej's results [27], there exists a smooth solution $u_{\varepsilon, j}$ to the equation

$$(3.25) \quad (\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{\varepsilon, j})^n = e^{-F_0 - \beta \varphi_{0, j} + \hat{C}} \frac{\omega_0^n}{(\varepsilon^2 + |s|_h^2)^{(1-\beta)}},$$

and $u_{\varepsilon, j}$ satisfies

$$(3.26) \quad \|u_{\varepsilon, j}\|_{L^\infty(M)} \leq C,$$

where \hat{C} is a uniform normalization constant independent of ε and j , constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β and F_0 .

We define function

$$(3.27) \quad \psi_{\varepsilon, j}(t) = (1 - te^{\beta t})\varphi_{0, j} + te^{\beta t}u_{\varepsilon, j} + h(t)e^{\beta t},$$

where

$$(3.28) \quad \begin{aligned} h(t) &= -t\|\varphi_{0, j}\|_{L^\infty(M)} - t\|u_{\varepsilon, j}\|_{L^\infty(M)} + n(t \log t - t)e^{-\beta t} \\ &\quad + \beta n \int_0^t e^{-\beta s} s \log s ds - \frac{\hat{C}}{\beta} e^{-\beta t} + \frac{\hat{C}}{\beta} \end{aligned}$$

and $h(0) = 0$.

Straightforward calculations show that

$$\begin{aligned}
\frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) - \beta \psi_{\varepsilon,j}(t) &= -\beta \varphi_{0,j} - e^{\beta t} \varphi_{0,j} + e^{\beta t} u_{\varepsilon,j} + e^{\beta t} \frac{\partial}{\partial t} h(t) \\
&= -\beta \varphi_{0,j} - e^{\beta t} \varphi_{0,j} + e^{\beta t} u_{\varepsilon,j} - e^{\beta t} \|\varphi_{0,j}\|_{L^\infty(M)} - e^{\beta t} \|u_{\varepsilon,j}\|_{L^\infty(M)} \\
&\quad + n \log t - \beta n(t \log t - t) + \beta n t \log t + \hat{C} \\
&\leq -\beta \varphi_{0,j} + n \log t + n \beta t + \hat{C}.
\end{aligned}$$

Therefore, we have

$$e^{\frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) - \beta \psi_{\varepsilon,j}(t)} \omega_0^n \leq t^n e^{n \beta t} e^{-\beta \varphi_{0,j} + \hat{C}} \omega_0^n.$$

When t is sufficiently small,

$$\begin{aligned}
\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t) &= (1 - t e^{\beta t})(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j}) + t e^{\beta t}(\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{\varepsilon,j}) \\
&\geq t e^{\beta t}(\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{\varepsilon,j}).
\end{aligned}$$

Combining the above inequalities,

$$\begin{aligned}
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t))^n &\geq t^n e^{n \beta t} (\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{\varepsilon,j})^n \\
&= t^n e^{n \beta t} e^{-F_0 - \beta \varphi_{0,j} + \hat{C}} \frac{\omega_0^n}{(\varepsilon^2 + |s|_h^2)^{(1-\beta)}} \\
&\geq e^{-F_0 + \frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) - \beta \psi_{\varepsilon,j}(t)} \frac{\omega_0^n}{(\varepsilon^2 + |s|_h^2)^{(1-\beta)}}.
\end{aligned}$$

This equation is equivalent to

$$(3.29) \quad \begin{cases} \frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) \leq \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t))^n}{\omega_0^n} + \beta \psi_{\varepsilon,j}(t) \\ \quad + F_0 + \log(\varepsilon^2 + |s|_h^2)^{(1-\beta)}. \\ \psi_{\varepsilon,j}(0) = \varphi_{0,j} \end{cases}$$

Let $\tilde{\psi}_{\varepsilon,j}(t) = \varphi_{\varepsilon,j}(t) - \psi_{\varepsilon,j}(t)$, then

$$(3.30) \quad \begin{cases} \frac{\partial}{\partial t} \tilde{\psi}_{\varepsilon,j}(t) \geq \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t) + \sqrt{-1} \partial \bar{\partial} \tilde{\psi}_{\varepsilon,j}(t))^n}{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t))^n} + \beta \tilde{\psi}_{\varepsilon,j}(t). \\ \tilde{\psi}_{\varepsilon,j}(0) = 0 \end{cases}$$

By the similar arguments as that in the proof of Proposition 3.3, for any $(t, z) \in [0, T] \times M$,

$$(3.31) \quad \tilde{\psi}_{\varepsilon,j}(t, z) \geq 0,$$

That is, for any $(t, z) \in [0, T] \times M$

$$\begin{aligned}
\varphi_{\varepsilon,j}(t, z) - \varphi_{0,j}(z) &\geq -t e^{\beta t} \varphi_{0,j} + t e^{\beta t} u_{\varepsilon,j} + h(t) e^{\beta t} \\
(3.32) \quad &\geq -C(e^{\beta t} - 1) + h_1(t) e^{\beta t},
\end{aligned}$$

where $h_1(t) = -n(t \log t - t) e^{-\beta t} + \beta n \int_0^t e^{-\beta s} s \log s ds$, constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β and F_0 . Let $j \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we have

$$(3.33) \quad \varphi(t, z) - \varphi_0(z) \geq -C(e^{\beta t} - 1) + h_1(t) e^{\beta t}.$$

There exists δ_2 such that for any $t \in [0, \delta_2]$,

$$(3.34) \quad -C(e^{\beta t} - 1) + h_1(t) e^{\beta t} > -\epsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$, then for any $t \in (0, \delta]$ and $z \in M$,

$$(3.35) \quad -\epsilon < \varphi(t, z) - \varphi_0(z) < \epsilon.$$

This completes the proof of the proposition. \square

Theorem 3.6. $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ is a long-time solution to the conical Kähler-Ricci flow (1.5).

Proof: We should only prove that $\omega(t)$ satisfies equation (1.5) in the sense of currents on $[0, \infty) \times M$.

Let $\eta = \eta(t, x)$ be a smooth $(n-1, n-1)$ -form with compact support in $(0, \infty) \times M$. Without loss of generality, we assume that its compact support included in (δ, T) ($0 < \delta < T < \infty$). On $[\delta, T] \times M$, by (3.1), (3.2), (3.14) and (3.15), we know that $\log \frac{\omega_\varepsilon^n(t)(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega_0^n}$, $\log \frac{\omega^n(t)|s|_h^{2(1-\beta)}}{\omega_0^n}$, $\varphi_\varepsilon(t)$ and $\varphi(t)$ are uniformly bounded by constants depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , δ and T . On $[\delta, T]$, we have

$$\begin{aligned} & \int_M \frac{\partial \omega_\varepsilon(t)}{\partial t} \wedge \eta = \int_M \sqrt{-1} \partial \bar{\partial} \frac{\partial \varphi_\varepsilon(t)}{\partial t} \wedge \eta \\ &= \int_M \sqrt{-1} \partial \bar{\partial} \left(\log \frac{\omega_\varepsilon^n(t)(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta \varphi_\varepsilon(t) \right) \wedge \eta \\ &= \int_M \log \left(\log \frac{\omega_\varepsilon^n(t)(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta \varphi_\varepsilon(t) \right) \sqrt{-1} \partial \bar{\partial} \eta \\ &\xrightarrow{\varepsilon \rightarrow 0} \int_M \left(\log \frac{\omega^n(t)}{\omega_0^n} + F_0 + \beta \varphi(t) + \log |s|_h^{2(1-\beta)} \right) \sqrt{-1} \partial \bar{\partial} \eta \\ &= \int_M \sqrt{-1} \partial \bar{\partial} \left(\log \frac{\omega^n(t)}{\omega_0^n} + F_0 + \beta \varphi(t) + \log |s|_h^{2(1-\beta)} \right) \wedge \eta \\ (3.36) \quad &= \int_M (-Ric(\omega(t)) + \beta \omega(t) + 2\pi(1-\beta)[D]) \wedge \eta. \end{aligned}$$

At the same time, there also holds

$$\begin{aligned} \int_M \omega_{\varphi_\varepsilon(t)} \wedge \frac{\partial \eta}{\partial t} &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon(t) \wedge \frac{\partial \eta}{\partial t} \\ &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi_\varepsilon(t) \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\ &\xrightarrow{\varepsilon_i \rightarrow 0} \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi(t) \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\ &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \sqrt{-1} \partial \bar{\partial} \varphi(t) \frac{\partial \eta}{\partial t} \\ (3.37) \quad &= \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t}. \end{aligned}$$

On the other hand, $\varphi_\varepsilon(t)$ and $\frac{\partial \varphi_\varepsilon(t)}{\partial t}$ are uniformly bounded on $[\delta, T] \times M$, $\varphi(t)$ and $\frac{\partial \varphi(t)}{\partial t}$ are uniformly bounded on $[\delta, T] \times (M \setminus D)$, therefore

$$\begin{aligned} \frac{\partial}{\partial t} \int_M \omega_{\varphi_\varepsilon(t)} \wedge \eta &= \int_M \varphi_\varepsilon(t) \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\ &\quad + \int_M \frac{\partial \varphi_\varepsilon(t)}{\partial t} \sqrt{-1} \partial \bar{\partial} \eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} \end{aligned}$$

$$\begin{aligned}
& \xrightarrow{\varepsilon \rightarrow 0} \int_M \varphi(t) \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\
& + \int_M \frac{\partial \varphi}{\partial t} \sqrt{-1} \partial \bar{\partial} \eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} \\
(3.38) \quad & = \frac{\partial}{\partial t} \int_M \omega(t) \wedge \eta.
\end{aligned}$$

Combining equality

$$\frac{\partial}{\partial t} \int_M \omega_{\varphi_\varepsilon(t)} \wedge \eta = \int_M \frac{\partial \omega_{\varphi_\varepsilon(t)}}{\partial t} \wedge \eta + \int_M \omega_{\varphi_\varepsilon(t)} \wedge \frac{\partial \eta}{\partial t}$$

with equalities (3.36)-(3.38), on $[\delta, T]$, we have

$$\begin{aligned}
\frac{\partial}{\partial t} \int_M \omega(t) \wedge \eta &= \int_M (-Ric(\omega(t)) + \beta \omega(t) + 2\pi(1-\beta)[D]) \wedge \eta \\
(3.39) \quad &+ \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t}.
\end{aligned}$$

Integrating from 0 to ∞ on both sides,

$$\begin{aligned}
\int_{M \times (0, \infty)} \frac{\partial \omega(t)}{\partial t} \wedge \eta \, dt &= - \int_{M \times (0, \infty)} \omega(t) \wedge \frac{\partial \eta}{\partial t} \, dt = - \int_0^\infty \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t} \, dt \\
&= \int_0^\infty \int_M (-Ric(\omega(t)) + \beta \omega(t) + 2\pi(1-\beta)[D]) \wedge \eta \, dt \\
&= \int_{M \times (0, \infty)} (-Ric(\omega(t)) + \beta \omega(t) + 2\pi(1-\beta)[D]) \wedge \eta \, dt.
\end{aligned}$$

By the arbitrariness of η , we prove that $\omega(t)$ satisfies the conical Kähler-Ricci flow (1.5) in the sense of currents on $(0, \infty) \times M$. \square

Now we are ready to prove the uniqueness of the parabolic Monge-Ampère equation (3.13) starting with $\varphi_0 \in \mathcal{E}_p(M, \omega_0)$ for some $p > 1$.

Theorem 3.7. *Let $\varphi_i(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D))$ ($i = 1, 2$) be two long-time solutions to the parabolic Monge-Ampère equation*

$$(3.40) \quad \frac{\partial \varphi_i(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_i(t))^n}{\omega_0^n} + F_0 + \beta \varphi_i(t) + \log |s|_h^{2(1-\beta)}$$

on $(0, \infty) \times (M \setminus D)$. If φ_i ($i = 1, 2$) satisfy

- For any $0 < \delta < T < \infty$, there exists uniform constant C such that

$$C^{-1} \omega_\beta \leq \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_i(t) \leq C \omega_\beta$$

on $[\delta, T] \times (M \setminus D)$;

- On $[\delta, T]$, there exist constant $\alpha > 0$ and C^* such that $\varphi_i(t)$ is C^α on M with respect to ω_0 and $\|\frac{\partial \varphi_i(t)}{\partial t}\|_{L^\infty(M \setminus D)} \leq C^*$;
- $\lim_{t \rightarrow 0^+} \|\varphi_i(t) - \varphi_0\|_{L^\infty(M)} = 0$.

Then $\varphi_1 = \varphi_2$.

Proof: We apply Jeffres' trick [25] in the parabolic case. For any $0 < t_1 < T < \infty$ and $a > 0$. Let $\phi_1(t) = \varphi_1(t) + a|s|_h^{2q}$, where $0 < q < 1$ is determined later. The

evolution of ϕ_1 is

$$\frac{\partial \phi_1(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_1(t))^n}{\omega_0^n} + F_0 + \beta \phi_1(t) - a\beta |s|_h^{2q} + \log |s|_h^{2(1-\beta)}.$$

Denote $\psi(t) = \phi_1(t) - \varphi_2(t)$ and $\hat{\Delta} = \int_0^1 g_{s\varphi_1+(1-s)\varphi_2}^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} ds$, $\psi(t)$ evolves along the following equation

$$\frac{\partial \psi(t)}{\partial t} = \hat{\Delta} \psi(t) - a\hat{\Delta} |s|_h^{2q} + \beta \psi(t) - a\beta |s|_h^{2q}.$$

By the equivalence of the metrics and the equation

$$\sqrt{-1} \partial \bar{\partial} |s|_h^{2q} = q^2 |s|_h^{2q} \sqrt{-1} \partial \log |s|_h^2 \wedge \bar{\partial} \log |s|_h^2 + q |s|_h^{2q} \sqrt{-1} \partial \bar{\partial} \log |s|_h^2,$$

we obtain the estimate

$$\begin{aligned} \hat{\Delta} |s|_h^{2q} &\geq q |s|_h^{2q} g_{s\varphi_1+(1-s)\varphi_2}^{i\bar{j}} \left(\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log |s|_h^2 \right) \\ &= -q |s|_h^{2q} g_{s\varphi_1+(1-s)\varphi_2}^{i\bar{j}} g_{0,i\bar{j}} \\ &\geq -C q |s|_h^{2q} g_{\beta}^{i\bar{j}} g_{0,i\bar{j}} \\ &\geq -C \end{aligned}$$

on $M \setminus D$, where constant C independent of a , and we apply the fact that $\omega_{\beta} \geq \gamma \omega_0$ on $M \setminus D$ for some constant γ . Then we obtain

$$\frac{\partial \psi(t)}{\partial t} \leq \hat{\Delta} \psi(t) + \beta \psi(t) + aC.$$

Let $\tilde{\psi} = e^{-\beta(t-t_1)} \psi + \frac{aC}{\beta} e^{-\beta(t-t_1)} - \epsilon(t-t_1)$. By choosing suitable $0 < q < 1$, we can assume that the space maximum of $\tilde{\psi}$ on $[t_1, T] \times M$ is attained away from D . Let (t_0, x_0) be the maximum point. If $t_0 > t_1$, by the maximum principle, at (t_0, x_0) , we have

$$0 \leq \left(\frac{\partial}{\partial t} - \hat{\Delta} \right) \tilde{\psi}(t) \leq -\epsilon,$$

which is impossible, hence $t_0 = t_1$. Then for $(t, x) \in [t_1, T] \times M$, we obtain

$$\begin{aligned} \psi(t, x) &\leq e^{\beta T} \|\varphi_1(t_1, x) - \varphi_2(t_1, x)\|_{L^\infty(M)} \\ &\quad + aC e^{\beta T} + \epsilon T e^{\beta T} \end{aligned}$$

Let $a \rightarrow 0$ and then $t_1 \rightarrow 0^+$, we get

$$\varphi_1(t) - \varphi_2(t) \leq \epsilon T e^{\beta T}.$$

It shows that $\varphi_1(t) \leq \varphi_2(t)$ after we let $\epsilon \rightarrow 0$. By the same reason we have $\varphi_2(t) \leq \varphi_1(t)$, then we prove that $\varphi_1(t) = \varphi_2(t)$. \square

Lemma 3.8. *Assume that on $(0, \infty) \times (M \setminus D)$, $f(t, x)$ is a smooth function and satisfies $\sqrt{-1} \partial \bar{\partial} f(t, x) = 0$. If*

$$(3.41) \quad \|f(t, x)\|_{L^\infty([\delta, T] \times (M \setminus D))} \leq C_{\delta, T},$$

where $0 < \delta < T < \infty$. Then $f(t, x) = f(t)$, i.e. function $f(t, x)$ depends only on t on $(0, \infty) \times (M \setminus D)$.

Proof: We prove this lemma by the logarithmic cutoff trick. Fix a cutoff function $\eta : [0, \infty) \rightarrow [0, 1]$ such that $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ for $s > 2$. We define

$$(3.42) \quad \gamma(x) = \eta\left(\frac{\log r}{\log \varepsilon}\right),$$

where r is the the distance function from the divisor. Then $\gamma = 0$ for $r < \varepsilon^2$ and $\gamma = 1$ for $r > \varepsilon$. Straightforward calculations shows that

$$\begin{aligned} \int_M |\partial\gamma|_{g_0}^2 dV_0 &= \int_M (\eta')^2 \frac{1}{r^2 (\log \varepsilon)^2} |\partial r|_{g_0}^2 dV_0 \\ &\leq C \int_{\varepsilon^2}^{\varepsilon} \frac{1}{r^2 (\log \varepsilon)^2} r dr = -C \frac{1}{\log \varepsilon}, \end{aligned}$$

so we have $\int_M |\partial\gamma|_{g_0}^2 dV_0 \rightarrow 0$ as $\varepsilon \rightarrow 0$. On the other hand, when $t \in [\delta, T]$,

$$\begin{aligned} 0 &= \int_M \operatorname{div}_{g_0} (\gamma^2 f (\bar{\nabla}_{g_0} f)) dV_0 \\ &= 2 \int_M \gamma f \langle \partial\gamma, \bar{\partial}f \rangle dV_0 + \int_M \gamma^2 |\partial f|_{g_0}^2 dV_0 \\ &\geq \frac{1}{2} \int_M \gamma^2 |\partial f|_{g_0}^2 dV_0 - 2 \int_M f^2 |\partial\gamma|_{g_0}^2 dV_0, \end{aligned}$$

which implies

$$(3.43) \quad \int_M \gamma^2 |\partial f|_{g_0}^2 dV_0 \leq C \int_M |\partial\gamma|_{g_0}^2 dV_0,$$

where constant C depends only on $\|f\|_{L^\infty([\delta, T] \times (M \setminus D))}$. Passing to the limit and then combining with the arbitrary choice of δ and T , we obtain

$$(3.44) \quad \int_M |\partial f|_{g_0}^2 dV_0 = 0 \quad \text{on } (0, \infty).$$

Hence function $f(t, x)$ depends only on t . □

Theorem 3.9. $\omega_{\varphi(t)} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ is the unique long-time solution to the conical Kähler-Ricci flow (1.5).

Proof: Suppose there is another solution $\omega_{\phi(t)} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t)$ to the conical Kähler-Ricci flow (1.5). It follows from Lemma 3.8 that

$$(3.45) \quad \frac{\partial\phi(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t))^n}{\omega_0^n} + F_0 + \beta\phi(t) + \log |s|_h^{2(1-\beta)} + f(t)$$

on $(0, \infty) \times (M \setminus D)$ for a smooth function $f(t)$ defined on $(0, \infty)$, and $\phi(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D))$ satisfies

- For any $0 < \delta < T < \infty$, there exists uniform constant C such that

$$C^{-1}\omega_\beta \leq \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t) \leq C\omega_\beta \quad \text{on } [\delta, T] \times (M \setminus D);$$

- On $[\delta, T]$, there exist constant $\alpha > 0$ and C such that $\phi(t)$ is C^α on M with respect to ω_0 and $\|\frac{\partial\phi(t)}{\partial t}\|_{L^\infty(M \setminus D)} \leq C$;
- $\lim_{t \rightarrow 0^+} \|\phi(t) - \varphi_0\|_{L^\infty(M)} = 0$.

For any $0 < t_1 < T < \infty$ and $a > 0$. Let $\psi(t) = \phi(t) + a|s|_h^{2q} - \varphi(t)$, where $0 < q < 1$ is determined later. Then

$$\frac{\partial \psi(t)}{\partial t} = \hat{\Delta} \psi(t) - a \hat{\Delta} |s|_h^{2q} + \beta \psi(t) - a \beta |s|_h^{2q} + f(t).$$

By the same arguments as that in the proof of Proposition 3.7, for any $(t, x) \in [t_1, T] \times M$, we have

$$\begin{aligned} \psi(t, x) &\leq e^{\beta(t-t_1)} \|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} \\ &\quad + a C e^{\beta(t-t_1)} + \epsilon(t-t_1) e^{\beta(t-t_1)} \\ &\quad + e^{\beta(t-t_1)} \int_{t_1}^t e^{-\beta(s-t_1)} f(s) ds \end{aligned}$$

Let $a \rightarrow 0$, we obtain

$$\begin{aligned} \phi(t) - \varphi(t) &\leq e^{\beta(t-t_1)} \|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} \\ &\quad + \epsilon(t-t_1) e^{\beta(t-t_1)} + e^{\beta(t-t_1)} \int_{t_1}^t e^{-\beta(s-t_1)} f(s) ds. \end{aligned}$$

By the similar arguments, we can obtain

$$\begin{aligned} \varphi(t) - \phi(t) &\leq e^{\beta(t-t_1)} \|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} \\ &\quad + \epsilon(t-t_1) e^{\beta(t-t_1)} - e^{\beta(t-t_1)} \int_{t_1}^t e^{-\beta(s-t_1)} f(s) ds. \end{aligned}$$

Therefore, for any $t > t_1 > 0$, we have

$$\begin{aligned} \inf_M (\phi(t) - \varphi(t)) &\geq \sup_M (\phi(t) - \varphi(t)) - 2e^{\beta(t-t_1)} \|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} \\ &\quad - 2\epsilon(t-t_1) e^{\beta(t-t_1)} \\ &\geq \sup_M (\phi(t) - \varphi(t)) - 2e^{\beta T} \|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} \\ &\quad - 2\epsilon T e^{\beta T}. \end{aligned}$$

Let $t_1 \rightarrow 0^+$ and then $\epsilon \rightarrow 0$, we conclude that $\phi(t) = \varphi(t) + e^{\beta t} \int_0^t e^{-\beta s} f(s) ds$. Then $\omega_{\phi(t)} = \omega_{\varphi(t)}$ on $(0, \infty) \times (M \setminus D)$. \square

4. THE CONVERGENCE OF THE CONICAL KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA

In this section, we study the convergence of the conical Kähler-Ricci flow (1.5) on Fano manifold with positive twisted first Chern class. Our discussion is very similar as that in [34], but we need new arguments on estimates of the twisted Ricci potential $u_\varepsilon(t)$ and the term $|\dot{\varphi}_\varepsilon|$ when we handle the weak initial data case.

Without loss of generality, we assume $\lambda = 1$ (i.e. $\mu = \beta$). We first prove the uniform Perelman's estimates along the twisted Kähler-Ricci flow

$$(4.1) \quad \begin{cases} \frac{\partial \omega_\varepsilon(t)}{\partial t} = -Ric(\omega_\varepsilon(t)) + \beta \omega_\varepsilon(t) + \theta_\varepsilon. \\ \omega_\varepsilon(t)|_{t=0} = \omega_{\varphi_0} \end{cases}$$

By the same argument as Proposition 4.1 in [34], we have

Proposition 4.1. $t^2(R(g_{\varepsilon,j}(t)) - \text{tr}_{g_{\varepsilon,j}(t)}\theta_\varepsilon)$ is uniformly bounded from below along the twisted Kähler-Ricci flow (2.3), i.e. there exists a uniform constant C , such that

$$(4.2) \quad t^2(R(g_{\varepsilon,j}(t)) - \text{tr}_{g_{\varepsilon,j}(t)}\theta_\varepsilon) \geq -C$$

for any $t \geq 0$, $j \in \mathbb{N}^+$ and $\varepsilon > 0$, while the constant C only depends on β and n . In particular,

$$(4.3) \quad R(g_{\varepsilon,j}(t)) - \text{tr}_{g_{\varepsilon,j}(t)}\theta_\varepsilon \geq -C$$

when $t \geq \frac{1}{2}$.

Remark 4.2. By Proposition 2.5, we know that there exists constant C only depending on β and n , such that

$$(4.4) \quad R(g_\varepsilon(t)) - \text{tr}_{g_\varepsilon(t)}\theta_\varepsilon \geq -C$$

along the twisted Kähler-Ricci flow (4.1) for any $\varepsilon > 0$ when $t \geq \frac{1}{2}$.

Straightforward calculation shows that the twisted Ricci potential $u_\varepsilon(t)$ with respect to $\omega_\varepsilon(t)$ at $t = \frac{1}{2}$ can be written as

$$(4.5) \quad u_\varepsilon\left(\frac{1}{2}\right) = \log \frac{\omega_\varepsilon^n\left(\frac{1}{2}\right)(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta\varphi_\varepsilon\left(\frac{1}{2}\right) + C_{\varepsilon,\frac{1}{2}},$$

where $C_{\varepsilon,\frac{1}{2}}$ is a normalization constant such that $\frac{1}{V} \int_M e^{-u_\varepsilon(\frac{1}{2})} dV_{\varepsilon,\frac{1}{2}} = 1$. By (3.1) and (3.2), we conclude that $C_{\varepsilon,\frac{1}{2}}$ and $u_\varepsilon(\frac{1}{2})$ are uniformly bounded. Let $a_\varepsilon(t) = \frac{\beta}{V} \int_M u_\varepsilon(t) e^{-u_\varepsilon(t)} dV_{\varepsilon,t}$, then by Lemma 4.4 in [34], we have

Lemma 4.3. *There exists a uniform constant C , such that*

$$(4.6) \quad |a_\varepsilon(t)| \leq C$$

for any $t \geq \frac{1}{2}$ and $\varepsilon > 0$.

Now we consider the twisted Kähler-Ricci flows (4.1) starting at $t = \frac{1}{2}$. Using the estimates (4.4), (4.6) and following the arguments in section 4 of [34], we have the following uniform Perelman's estimates.

Theorem 4.4. *Let $g_\varepsilon(t)$ be a solution of the twisted Kähler Ricci flow, i.e. the corresponding form $\omega_\varepsilon(t)$ satisfies the equation (4.1) with initial metric ω_{φ_0} , $u_\varepsilon(t) \in C^\infty((0, \infty) \times M)$ is the twisted Ricci potential satisfying*

$$(4.7) \quad -\text{Ric}(\omega_\varepsilon(t)) + \beta\omega_\varepsilon(t) + \theta_\varepsilon = \sqrt{-1}\partial\bar{\partial}u_\varepsilon(t)$$

and $\frac{1}{V} \int_M e^{-u_\varepsilon(t)} dV_{\varepsilon,t} = 1$, where $\theta_\varepsilon = (1 - \beta)(\omega_0 + \sqrt{-1}\partial\bar{\partial}\log(\varepsilon^2 + |s|_h^2))$. Then for any $\beta \in (0, 1)$, there exists a uniform constant C , such that

$$(4.8) \quad |R(g_\varepsilon(t)) - \text{tr}_{g_\varepsilon(t)}\theta_\varepsilon| \leq C,$$

$$(4.9) \quad \|u_\varepsilon(t)\|_{C^1(g_\varepsilon(t))} \leq C,$$

$$(4.10) \quad \text{diam}(M, g_\varepsilon(t)) \leq C$$

hold for any $t \geq 1$ and $\varepsilon > 0$, where $R(g_\varepsilon(t)) - \text{tr}_{g_\varepsilon(t)}\theta_\varepsilon$ and $\text{diam}(M, g_\varepsilon(t))$ are the twisted scalar curvature and diameter of the manifold respectively with respect to the metric $g_\varepsilon(t)$.

If $\varphi_\varepsilon(t)$ is a solution to the Monge-Ampère equation

$$(4.11) \quad \frac{\partial \varphi_\varepsilon(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon(t))^n}{\omega_0^n} + F_0 + \beta \varphi_\varepsilon(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta}$$

on $(0, \infty) \times M$ with initial value $\varphi_\varepsilon(0) = \varphi_0$, it is obvious that $\phi_\varepsilon(t) = \varphi_\varepsilon(t) + C e^{\beta t}$ is a solution to equation (4.11) with initial value $\phi_\varepsilon(0) = \varphi_0 + C$. At the same time, $\omega_{\phi_\varepsilon(t)} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_\varepsilon(t)$ is also a solution to the twisted Kähler-Ricci flow (4.1) with initial value ω_{φ_0} .

From (3.1), we know that $\varphi_\varepsilon(t)$ is uniformly bounded on $[0, T] \times M$ by a constant C which depends only on $\|\varphi_0\|_{L^\infty(M)}$, β and T . Now, we consider the solution $\psi_\varepsilon(t) = \varphi_\varepsilon(t) + \tilde{C}_{\varepsilon,1} e^{\beta t}$ to the equation

$$(4.12) \quad \begin{cases} \frac{\partial \psi_\varepsilon(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_\varepsilon(t))^n}{\omega_0^n} + F_0 + \beta \psi_\varepsilon(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta} \\ \psi_\varepsilon(0) = \varphi_0 + \tilde{C}_{\varepsilon,1} \end{cases} \quad \text{on } (0, \infty) \times M,$$

where

$$\tilde{C}_{\varepsilon,1} = e^{-\beta} \frac{1}{\beta} \left(\int_1^{+\infty} e^{-\beta t} \|\nabla u_\varepsilon(t)\|_{L^2}^2 dt - \frac{1}{V} \int_M (F_{\varepsilon,1} + \beta \varphi_\varepsilon(1)) dV_{\varepsilon,1} \right),$$

$F_{\varepsilon,1} = F_0 + \log(\frac{\omega_\varepsilon(1)^n}{\omega_0^n} \cdot (\varepsilon^2 + |s|_h^2)^{1-\beta})$ and $dV_{\varepsilon,1} = \frac{\omega_\varepsilon^n(1)}{n!}$. By (3.1), (3.2) and the above uniform Perelman's estimates (4.9), we know that the constant $\tilde{C}_{\varepsilon,1}$ is well-defined and uniformly bounded. Straightforward calculation shows that the twisted Ricci potential $u_\varepsilon(1)$ with respect to $\omega_\varepsilon(1)$ can be written as

$$(4.13) \quad u_\varepsilon(1) = \log \frac{\omega_\varepsilon^n(1)(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta \varphi_\varepsilon(1) + C_{\varepsilon,1},$$

where $C_{\varepsilon,1}$ is a normalization constant such that $\frac{1}{V} \int_M e^{-u_\varepsilon(1)} dV_{\varepsilon,1} = 1$. Then

$$(4.14) \quad C_{\varepsilon,1} = \log \left(\frac{1}{V} \int_M e^{-F_0 - \beta \varphi_\varepsilon(1)} \frac{dV_0}{(\varepsilon^2 + |s|_h^2)^{1-\beta}} \right).$$

By (3.1) and (3.2), we conclude that $C_{\varepsilon,1}$ and $u_\varepsilon(1)$ are uniformly bounded.

Let $u_\varepsilon(t) = \dot{\psi}_\varepsilon(t) + c_\varepsilon(t)$. By equation (4.12) and equality (4.13), we have

$$(4.15) \quad c_\varepsilon(1) = C_{\varepsilon,1} - \beta e^\beta \tilde{C}_{\varepsilon,1}.$$

Proposition 4.5. *There exists a uniform constant C such that*

$$\|\dot{\psi}_\varepsilon(t)\|_{C^0} \leq C$$

for any $\varepsilon > 0$ and $t \geq 1$.

Proof: As in [37], when $t \geq 1$, we let

$$(4.16) \quad \alpha_\varepsilon(t) = \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t) dV_{\varepsilon,t} = \frac{1}{V} \int_M u_\varepsilon(t) dV_{\varepsilon,t} - c_\varepsilon(t).$$

Through computing, we have

$$(4.17) \quad \frac{d}{dt} \alpha_\varepsilon(t) = \beta \alpha_\varepsilon(t) - \|\nabla \dot{\psi}_\varepsilon\|_{L^2}^2,$$

$$\begin{aligned}
e^{-\beta(t-1)}\alpha_\varepsilon(t) &= \alpha_\varepsilon(1) - \int_1^t e^{-\beta(s-1)} \|\nabla \dot{\psi}_\varepsilon\|_{L^2}^2 ds \\
(4.18) \quad &= \frac{1}{V} \int_M u_\varepsilon(1) dV_{\varepsilon,1} - c_\varepsilon(1) - \int_1^t e^{-\beta(s-1)} \|\nabla \dot{\psi}_\varepsilon\|_{L^2}^2 ds.
\end{aligned}$$

Putting (4.13) and (4.15) into (4.18), we have

$$(4.19) \quad e^{-\beta(t-1)}\alpha_\varepsilon(t) = \int_t^{+\infty} e^{-\beta(s-1)} \|\nabla \dot{\psi}_\varepsilon\|_{L^2}^2 ds.$$

By Theorem 4.4, we conclude that

$$(4.20) \quad 0 \leq \alpha_\varepsilon(t) = \int_t^{+\infty} e^{\beta(t-s)} \|\nabla \dot{\psi}_\varepsilon\|_{L^2}^2 ds \leq C.$$

Then we conclude that $\dot{\psi}_\varepsilon(t)$ is uniformly bounded by the uniform Perelman's estimates when $t \geq 1$. \square

We recall Aubin's functionals, Ding's functional and the twisted Mabuchi \mathcal{K} -energy functional.

$$\begin{aligned}
(4.21) \quad I_{\omega_0}(\phi) &= \frac{1}{V} \int_M \phi(dV_0 - dV_\phi), \\
J_{\omega_0}(\phi) &= \frac{1}{V} \int_0^1 \int_M \dot{\phi}_t(dV_0 - dV_{\phi_t}) dt,
\end{aligned}$$

where ϕ_t is a path with $\phi_0 = c$, $\phi_1 = \phi$.

$$(4.22) \quad F_{\omega_0}^0(\phi) = J_{\omega_0}(\phi) - \frac{1}{V} \int_M \phi dV_0,$$

$$(4.23) \quad F_{\omega_0, \theta}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V} \int_M \phi dV_0 - \frac{1}{\beta} \log\left(\frac{1}{V} \int_M e^{-u_{\omega_0} - \beta\phi} dV_0\right),$$

$$\begin{aligned}
(4.24) \quad \mathcal{M}_{\omega_0, \theta}(\phi) &= -\beta(I_{\omega_0}(\phi) - J_{\omega_0}(\phi)) - \frac{1}{V} \int_M u_{\omega_0}(dV_0 - dV_\phi) \\
&\quad + \frac{1}{V} \int_M \log \frac{\omega_\phi^n}{\omega_0^n} dV_\phi,
\end{aligned}$$

where u_{ω_0} is the twisted Ricci potential of ω_0 , i.e. $-Ric(\omega_0) + \beta\omega_0 + \theta = \sqrt{-1}\partial\bar{\partial}u_{\omega_0}$ and $\frac{1}{V} \int_M e^{-u_{\omega_0}} dV_{\omega_0} = 1$.

Proposition 4.6. *For any $t \geq 1$, the solution $\psi_\varepsilon(t)$ to equation (4.12) satisfies:*

$$\begin{aligned}
(i) \quad &\mathcal{M}_{\omega_0, \theta_\varepsilon}(\psi_\varepsilon(t)) - \beta F_{\omega_0}^0(\psi_\varepsilon(t)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t) dV_{\varepsilon,t} = C_\varepsilon, \\
(ii) \quad &\mathcal{M}_{\omega_0, \theta_\varepsilon}(\psi_\varepsilon(1)) \text{ is uniformly bounded,}
\end{aligned}$$

where C_ε in (i) can be bounded by a uniform constant C .

Proof: Following the argument in [34], since

$$(4.25) \quad \frac{d}{dt}(\mathcal{M}_{\omega_0, \theta_\varepsilon}(\psi_\varepsilon(t)) - \beta F_{\omega_0}^0(\psi_\varepsilon(t)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t) dV_{\varepsilon,t}) = 0,$$

we obtain that

$$\begin{aligned}
& \mathcal{M}_{\omega_0, \theta_\varepsilon}(\psi_\varepsilon(t)) - \beta F_{\omega_0}^0(\psi_\varepsilon(t)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t) dV_{\varepsilon,t} \\
&= \mathcal{M}_{\omega_1, \theta_\varepsilon}(\psi_\varepsilon(1)) - \beta F_{\omega_0}^0(\psi_\varepsilon(1)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(1) dV_{\varepsilon,1} \\
&= \frac{1}{V} \int_M \log \frac{\omega_\varepsilon^n(1)(|s|_h^2 + \varepsilon^2)^{1-\beta}}{e^{-F_0} \omega_0^n} dV_{\varepsilon,1} + \frac{\beta}{V} \int_M \psi_\varepsilon(1) dV_{\varepsilon,1} \\
&\quad - \frac{1}{V} \int_M F_0 + \log(|s|_h^2 + \varepsilon^2)^{1-\beta} dV_0 - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(1) dV_{\varepsilon,1}.
\end{aligned}$$

where the last equality can be bounded by a uniform constant. This gives a proof of (i). Furthermore, by the definition of $\mathcal{M}_{\omega_0, \theta_\varepsilon}$, we have

$$\begin{aligned}
\mathcal{M}_{\omega_0, \theta_\varepsilon}(\psi_\varepsilon(1)) &= \frac{1}{V} \int_M \log \frac{\omega_\varepsilon^n(1)(|s|_h^2 + \varepsilon^2)^{1-\beta}}{e^{-F_0} \omega_0^n} dV_{\varepsilon,1} - \beta I_{\omega_0}(\psi_\varepsilon(1)) + \beta J_{\omega_0}(\psi_\varepsilon(1)) \\
&\quad - \frac{1}{V} \int_M F_0 + \log(|s|_h^2 + \varepsilon^2)^{1-\beta} dV_0.
\end{aligned}$$

Since $I_{\omega_0}(\psi_\varepsilon(1))$ is uniformly bounded and $\frac{1}{n} J_{\omega_0} \leq \frac{1}{n+1} I_{\omega_0} \leq J_{\omega_0}$, we prove (ii). \square

Using Proposition 4.5 and 4.6, by following the arguments in section 5 of [34], we obtain the following uniform C^0 estimate of $\psi_\varepsilon(t)$ along the equation (4.12) under the assumption that the twisted Mabuchi \mathcal{K} -energy functional $\mathcal{M}_{\omega_0, \theta_\varepsilon}$ is uniformly proper on the space

$$(4.26) \quad \mathcal{H}(\omega_0) = \{\phi \in C^\infty(M) \mid \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi > 0\}.$$

Theorem 4.7. *Let $\psi_\varepsilon(t)$ be a solution of the flow (4.12). If the twisted Mabuchi \mathcal{K} -energy functional $\mathcal{M}_{\omega_0, \theta_\varepsilon}$ is uniformly proper on $\mathcal{H}(\omega_0)$, i.e. there exists a uniform function f such that*

$$(4.27) \quad \mathcal{M}_{\omega_0, \theta_\varepsilon}(\phi) \geq f(J_{\omega_0}(\phi))$$

for any ε and $\phi \in \mathcal{H}(\omega_0)$, where $f(t) : \mathbb{R}^+ \rightarrow \mathbb{R}$ is some monotone increasing function satisfying $\lim_{t \rightarrow +\infty} f(t) = +\infty$, then there exists a uniform constant C such that for any $\varepsilon > 0$ and $t \geq 0$

$$(4.28) \quad \|\psi_\varepsilon(t)\|_{C^0} \leq C.$$

We study the flow (4.1) start at $t = \frac{1}{2}$. We obtain $C^{-1} \omega_\varepsilon \leq \omega_\varepsilon(\frac{1}{2}) \leq C \omega_\varepsilon$ on M and $\|\psi_\varepsilon(\frac{1}{2})\|_{C^k(K)} \leq C_{k,K}$ on $K \subset\subset M \setminus D$ for some uniform constants C and $C_{k,K}$ in section 3, after getting the uniform bound of $\dot{\psi}_\varepsilon(t)$ and $\psi_\varepsilon(t)$, we can prove the uniform Laplacian C^2 estimates and local uniform estimates for any $t \geq 1$ and $\varepsilon > 0$ by the arguments in [34] (see Proposition 2.1 and 2.3 in [34]). In fact, we prove the following theorem.

Theorem 4.8. *Under the assumption in Theorem 4.7, for any $k \in \mathbb{N}^+$ and $K \subset\subset M \setminus D$, there exists constant $C_{k,K}$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, n , β , k , ω_0 and $\text{dist}_{\omega_0}(K, D)$, such that for any $\varepsilon > 0$ and $t \geq 1$, we have*

$$(4.29) \quad \|\psi_\varepsilon(t)\|_{C^k(K)} \leq C_{k,K}.$$

Now we assume that there exists a conical Kähler-Einstein metric with cone angle $2\pi\beta$ along D . When $\lambda > 0$ and there is no nontrivial holomorphic field which is tangent to D along D , Tian-Zhu [47] obtained the following Moser-Trudinger type inequality

$$(4.30) \quad F_{\omega_0, (1-\beta)D}(\phi) \geq \delta J_{\omega_0}(\phi) - C, \quad \forall \phi \in \mathcal{H}(\omega_0)$$

for some constants δ and C , where

$$F_{\omega_0, (1-\beta)D}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V} \int_M \phi dV_0 - \frac{1}{\beta} \log \left(\frac{1}{V} \int_M \frac{1}{|s|_h^{2(1-\beta)}} e^{-F_0 - \beta\phi} dV_0 \right)$$

is defined in [47] (see also [30]).

Remark 4.9. When $\lambda \geq 1$, R. Berman [1], Li-Sun [30] proved that there is no non-trivial holomorphic vector field on M tangent to divisor D , and Li-Sun also proved that the existence of conical Kähler-Einstein metric can deduce the properness of the Log Mabuchi \mathcal{K} -energy functional (see also Song-Wang's results in [43]).

By the definition of $F_{\omega_0, \theta_\varepsilon}$ and $F_{\omega_0, (1-\beta)D}$, we have

$$(4.31) \quad \begin{aligned} F_{\omega_0, \theta_\varepsilon}(\phi) - F_{\omega_0, (1-\beta)D}(\phi) &= \frac{1}{\beta} \log \left(\frac{1}{V} \int_M e^{-F_0 - C_0 - \beta\phi} \frac{dV_0}{|s|_h^{2(1-\beta)}} \right) \\ &\quad - \frac{1}{\beta} \log \left(\frac{1}{V} \int_M e^{-F_0 - C_\varepsilon - \beta\phi} \frac{dV_0}{(\varepsilon^2 + |s|_h^2)^{(1-\beta)}} \right) \\ &\geq -C, \end{aligned}$$

where C_0 and C_ε are two normalized constants, and C is a constant independent of ε . So the Ding's functional $F_{\omega_0, \theta_\varepsilon}$ is uniform proper. By the normalization and Jensen's inequality, we have

$$(4.32) \quad \frac{1}{V} \int_M -u_{\omega_\phi} dV_\phi \leq \log \left(\frac{1}{V} \int_M e^{-u_{\omega_\phi}} dV_\phi \right) = 0.$$

Then we have the following inequalities by (4.23), (4.24) and (4.32).

$$(4.33) \quad \begin{aligned} \mathcal{M}_{\omega_0, \theta_\varepsilon}(\phi) &= \beta F_{\omega_0, \theta_\varepsilon}(\phi) + \frac{1}{V} \int_M u_{\omega_\phi} dV_\phi - \frac{1}{V} \int_M u_{\omega_0} dV_0 \\ &\geq \beta F_{\omega_0, \theta_\varepsilon}(\phi) - \frac{1}{V} \int_M F_0 + C_\varepsilon + (1-\beta) \log(\varepsilon^2 + |s|_h^2) dV_0 \\ &\geq \beta F_{\omega_0, \theta_\varepsilon}(\phi) - C, \end{aligned}$$

where constant C independent of ε . Hence we deduce the uniform properness of the twisted Mabuchi \mathcal{K} -energy functional by (4.30), (4.31) and (4.33), i.e.

$$(4.34) \quad M_{\omega_0, \theta_\varepsilon}(\phi) \geq C_1 J_{\omega_0}(\phi) - C_2, \quad \forall \phi \in \mathcal{H}(\omega_0)$$

for some uniform constants C_1 and C_2 . At the same time, we have the uniqueness theorem of conical Kähler-Einstein metric (proved by B. Berndtsson in [2]) under the assumption that there is no nontrivial holomorphic field which is tangent to D . Using the above C^0 estimate and the uniqueness theorem, we can apply the arguments in section 6 of [34] to obtain the convergence result of the conical Kähler-Ricci flow (1.5), i.e. Theorem 1.3.

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JIAWEI LIU, BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA,

E-mail address: jwliu@math.pku.edu.cn

XI ZHANG, KEY LABORATORY OF WU WEN-TSUN MATHEMATICS, CHINESE ACADEMY OF SCIENCES, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, CHINA,

E-mail address: mathzx@ustc.edu.cn